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A monotonicity preserving, nonlinear, finite element upwind method for the transport equation

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ABSTRACT

We propose a simple upwind finite element method that is monotonicity preserving and weakly consistent of order $O(h^{\frac{3}{2}})$. The scheme is nonlinear, but since an explicit time integration method is used the added cost due to the nonlinearity is not prohibitive. We prove the monotonicity preserving property for the forward Euler method and for a second order Runge–Kutta method. The convergence properties of the Runge–Kutta finite element method are verified on a numerical example.

1. Introduction

The design of robust and accurate finite element methods for first order hyperbolic equations or convection dominated convection–diffusion problems remains an active field of research. Indeed the task of designing a numerical scheme that is of higher order than one in the zone where the exact solution is smooth, but preserves the monotonicity properties of the exact solution on the discrete level, is nontrivial. Since it is known that such a scheme necessarily must be nonlinear even for linear equations the typical strategy adopted when working with stabilized finite element methods is to add an additional nonlinear shock-capturing term, designed to make the method satisfy a discrete maximum principle [1–3]. These methods however often result in very ill-conditioned nonlinear equations and include parameters that may be difficult to tune and depend on the mesh geometry. Another approach is the flux corrected finite element method [4,5]. In this scheme the system matrix is manipulated so that it becomes a so called M-matrix, the inverse of which has positive coefficients which yields a maximum principle. This scheme is monotonicity preserving, but of first order. In order to improve the accuracy anti-diffusive mechanisms, or flux-limiter techniques, have been proposed that reduce the amount of dissipation in the smooth region by blending a low and a high order approximation [6,5,7].

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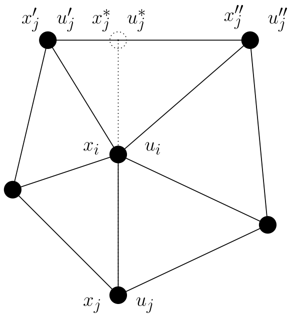

Fig. 1. Illustration of the macro patch Ω_i and the points x_i, x_j and x_j^* with associated function values u_i, u_j and u_j^*.

In this paper we will discuss a method that is related to both the above mentioned classes in the sense that the method consists of the addition of a nonlinear dissipative term to the standard Galerkin formulation as for a shock capturing term, but similarly as in a flux corrected transport method the nonlinear term uses the coefficients of the system matrix for its definition. The method is entirely derived from the finite element variational formulation and the guiding principle of the analysis has been to add the smallest perturbation to the centered standard Galerkin formulation that ensures that the method is monotonicity preserving. The salient features of the resulting method is that the optimal value of the stabilization parameter can be traced in the analysis, the monotonicity does not require any acute condition of the mesh and the artificial dissipation term depends on the residual of the exact solution in the form of a linear combination of the jumps of directional derivatives over each node (c.f. the edge based limiters that were proposed in the eighties, see [6] and references therein, but also [8,3]). Formally this leads to a method with $O(h^{\frac{3}{2}})$ artificial viscosity where the solution is smooth and we show in a numerical example that the expected $O(h^{\frac{3}{2}})$ convergence of the error in the L^2-norm, indeed holds after a suitable regularization of the stabilization term.

2. Model problem and finite element discretization

We consider the pure transport equation in $\mathbb{R}^2$

$$\partial_t u + \boldsymbol{\beta} \cdot \nabla u = 0 \tag{1}$$

with $u(x, 0) = u_0(x)$ where $u_0(x)$ is some function with compact support and $\boldsymbol{\beta} \in [W^{1,\infty}(\mathbb{R}^2)]^2$. Let $\mathcal{T}_h := \{K\}$ denote a conforming, shape regular, triangulation of $\mathbb{R}^2$. The finite element space of piecewise affine continuous functions is defined on $\mathcal{T}_h$ as

$$V_h := \{v_h \in H^1(\mathbb{R}^2) : v_h|_K \in P_1(K), \forall K \in \mathcal{T}_h\}$$

where $P_1(K)$ denotes the polynomials of degree less than or equal to 1 over K. The nodal basis functions of V_h will be denoted φ_i, i.e. $\varphi_i(x_j) = \delta_{ij}$, with δ_{ij} the Kronecker delta function. Any function $v_h \in V_h$ is then defined by $\sum_i v_i \varphi_i$, where the v_i denotes the nodal values of the function. We denote by $\mathcal{N}_K$ the set of indices of the vertices x_i, of K. We also introduce the length of the edge e_{ij} between the nodes x_i and x_j, $h_{ij} := |x_i - x_j|$ and the unit vector pointing from x_j to x_i, $\tau_{ij} := (x_i - x_j)/h_{ij}$. To each node x_i of the mesh we associate the macroelement $\Omega_i := \{K \in \mathcal{T}_h : x_i \in K\}$, with associated set of indices $\mathcal{N}_{\Omega_i}$ of the vertices $x_j \in \Omega_i$. For every node x_j in the boundary of Ω_i we associate a distance $h_{ij}^* > 0$ such that $x_j^* := x_i + h_{ij}^* \tau_{ij} \in \partial\Omega_i$ (see Fig. 1). The value of the finite element solution at x_j^* will be denoted $u_j^* := u_h(x_j^*)$. If u_j' and u_j'' denotes the values of u_h in the nodes of the endpoints of the edge with x_j^* in its interior we see that there exists some $\alpha_j^* \in (0, 1)$ such that $u_j^* = \alpha_j^* u_j' + (1 - \alpha_j^*) u_j''$. By the shape regularity assumption we know that the number of points x_j^* in the interior of any edge in Ω_i is upper bounded by some $n_i^* \in \mathbb{N}$. Let $\underline{h}_K$ denote the radius of the largest circle inscribed in a given triangle K, similarly let $\overline{h}_K$ denote the radius of the smallest circle circumscribing K. The