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Uniqueness and grow-up rate of solutions for pseudo-parabolic equations in $\mathbb{R}^n$ with a sublinear source

Sujin Khomrutai

Department of Mathematics and Computer Science, Faculty of Science, Chulalongkorn University, Bangkok 10330, Thailand

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ABSTRACT

In this paper, we solve an open problem appeared in Cao et al. (2009) concerning the uniqueness of solutions for a sublinear pseudo-parabolic Cauchy problem. In the zero initial case, we obtain the class of all non-trivial global solutions, whereas, the uniqueness of global solutions is established when the initial condition is non-zero. A lower grow-up rate of solutions is also obtained.

1. Introduction

In this paper, we consider solutions $u(x,t) \geq 0$ of the sublinear pseudo-parabolic Cauchy problem

$$\begin{cases} \partial_t u - \triangle \partial_t u = \triangle u + u^p & x \in \mathbb{R}^n, t > 0, \\ u(x,0) = u_0(x) \geq 0 & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $0 < p < 1$ is a constant and $n \geq 1$ is a positive integer. This problem was studied in [1] and the existence of global solutions was established within $C([0,\infty); C_b(\mathbb{R}^n))$. The question of uniqueness of solutions, however, has been left open. The purpose of this paper is to settle this question.

In recent years, there is a rich literature addressing the existence, or uniqueness of solutions for pseudo-parabolic problems in bounded, or unbounded domains, and for periodic solutions. Among many others, we mention [2,1,3–6]. From practical point of view, we should also mention [7–10], where pseudo-parabolic problems appear as models for porous media flows with or without dynamic capillarity.

Setting $u = e^{-t}U$ in (1.1), we get the nonlocal formulation: $\partial_t U = \mathcal{B}U + e^{(1-p)t}\mathcal{B}U^p$, $U|_{t=0} = u_0$ and upon integration we obtain the mild formulation:

E-mail address: sujin.k@chula.ac.th.

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$$U(x,t) = \mathcal{M}U(x,t) := u_0(x) + \int_0^t \mathcal{B}U(x,s)ds + \int_0^t e^{(1-p)s}\mathcal{B}U^p(x,s)ds. \tag{1.2}$$

$\mathcal{B} = (1-\triangle)^{-1}$ is the *Bessel potential operator* given by

$$\mathcal{B}\varphi = \int_{\mathbb{R}^n} B(x-y)\varphi(y)dy, \qquad B(x) = |x|^{(2-n)/2}K_{(n-2)/2}(|x|),$$

and K_ν is the modified Bessel function of the second kind.

Apart from $\mathcal{B}$, we also need the *Green operator* $\mathcal{G}(t) = e^{-t}e^{t\mathcal{B}} = e^{-t}\sum_{k=0}^{\infty}\frac{t^k}{k!}\mathcal{B}^k (t>0)$. Both $\mathcal{B}$ and $\mathcal{G}(t)$ are positive, bounded, linear operators on $C_b(\mathbb{R}^n)$. Note that $\mathcal{B}(1) = 1$ and $\mathcal{G}(t)(1) = 1$. More details can be found in [11,12].

Let us state the main definition in this work.

Definition 1. A mild solution (resp., super-solution, or sub-solution) of (1.1) is a function $u \in C([0,T); C_b(\mathbb{R}^n))$ for some $0 < T \le \infty$ such that

$$U(x,t) = \mathcal{M}U(x,t) \quad (\text{resp., } U \ge \mathcal{M}U, \text{ or } U \le \mathcal{M}U)$$

for all $x \in \mathbb{R}^n$, $t \in [0,T)$. If $T = \infty$, such a function u is called a global mild solution (resp., super-solution, or sub-solution). We note that $U = e^t u$.

In this work, we prove the following main results.

Theorem 1. *Let $u \ge 0$ be a mild super-solution of (1.1) and $0 \le u_0 \in C(\mathbb{R}^n)$ with $u_0 \not\equiv 0$. Then $u(x,t) \ge ((1-p)t)^q$ for all $(x,t) \in \mathbb{R}^n \times [0,T)$ where $q = 1/(1-p)$.*

Corollary 1. *If $u_0 \equiv 0$, then all the non-trivial global mild solutions of (1.1) have the form*

$$u(x,t) = ((1-p)(t-\tau)_+)^q \quad (\tau \ge 0). \tag{1.3}$$

Remark 1. It is straightforward to see that, the nontrivial mild solutions (1.3) are obtained by solving the ordinary differential equation emerging from (1.1), if assuming that u is only time dependent. Also observe that these solutions are just translations in time of the maximal solution $((1-p)t)^q$.

Theorem 2. *Let $u, v \in C([0,T); C_b(\mathbb{R}^n))$, $u, v \ge 0$, be mild super-solution and sub-solution, respectively, of (1.1), and $u_0, v_0 \in C^\alpha(\mathbb{R}^n)$ $(0 < \alpha < 1)$ satisfy $u_0(x) \ge v_0(x) \ge 0$, $u_0 \not\equiv 0$. Then $u \ge v$ on $\mathbb{R}^n \times [0,T)$.*

Corollary 2. *If $u_0 \in C^\alpha(\mathbb{R}^n)$ $(0 < \alpha < 1)$, $u_0 \ge 0$, and $u_0 \not\equiv 0$, then there exists a unique global mild solution u to the Cauchy problem (1.1).*

2. Lower bound of grow-up rate

Lemma 1. *Let $u = e^{-t}U \ge 0$ be a mild super-solution of (1.1) and $0 \le u_0 \in C(\mathbb{R}^n)$, $u_0 \not\equiv 0$. Then for $\delta > 1$, $t_0 \in (0,T)$, there is a constant $c_\delta = c(\delta, u_0, t_0)$ such that $\mathcal{M}U|_{t_0} \ge c_\delta e^{-\delta|x|}$.*

Proof. Since $\mathcal{B}$ is monotone and $U \ge 0$, we have $U \ge \mathcal{M}U \ge u_0$. Then $U \ge \mathcal{M}U \ge \int_0^t \mathcal{B}u_0(x)ds > 0$ on $\mathbb{R}^n \times (0,T)$. By the asymptotic behavior of $B(x)$ as $|x| \to \infty$ and $|x| \to 0$ [13], there is a constant $b > 0$ such that

$$B(x) \ge b\theta(x)e^{-|x|} \quad \text{where } \theta(x) = \begin{cases} |x|^{(1-n)/2} & \text{if } n \ne 2 \text{ or } |x| \ge 1, \\ 1 - \ln|x| & \text{if } n = 2 \text{ and } |x| < 1. \end{cases} \tag{2.1}$$