Applied Mathematics Letters 48 (2015) 41–46

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Large deviations for stochastic 3D cubic Ginzburg–Landau equation with multiplicative noise

Lian Yang [a], Xueke Pu [b,*]

[a] School of Mathematics and Physics, Chongqing University of Posts and Telecommunications, Chongqing 400065, PR China
[b] College of Mathematics and Statistics, Chongqing University, Chongqing 401331, PR China

ARTICLE INFO

Article history:
Received 23 December 2014
Received in revised form 17 February 2015
Accepted 17 February 2015
Available online 27 March 2015

Keywords:
Large deviations
Stochastic 3D Ginzburg–Landau equation
Laplace principle
Weak convergence method

ABSTRACT

This paper considers the large deviation principle for the stochastic 3D cubic Ginzburg–Landau equation perturbed by a small multiplicative noise. Using the weak convergence approach, we establish a large deviation principle of Freidlin–Wentzell type by proving a Laplace principle.

1. Introduction

In this paper, we consider the following stochastic 3D cubic Ginzburg–Landau equation

$$du = [(\lambda + i\alpha)\Delta u + \gamma u - (\kappa + i\beta)|u|^2 u]dt + \sum_{k=1}^{\infty} \sqrt{\varepsilon}\lambda_k u dW_t^k, \tag{1.1}$$

with zero-Dirichlet boundary condition

$$u(t, x)|_{x\in\partial D} = 0,$$

where $D \subset R^3$ is a bounded smooth domain, $u : D \to \mathbb{C}$ is the unknown complex-valued function and $\{W_t^k : t \geq 0, k = 1, 2, \ldots\}$ is a sequence of independent of one dimensional standard Brownian motions on some complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}; P)$. The purpose of the present paper is to establish the large deviation principle in the space $\mathcal{E}$ defined below for small noise for (1.1) with H^1-initial data $u(x, 0) = u_0(x)$.

The Ginzburg–Landau equation is an important model equation in superconductivity and is found to be one of the fundamental equations in modern physics, in particular, in the description of spatial pattern formation and the onset of instabilities in non-equilibrium fluid dynamical systems. Therefore it was broadly studied in recent years from different points of view [1–3]. In [2], we obtained the existence and pathwise uniqueness for H^1-solutions for (1.1) with slightly more general noise term, and proved the ergodicity for the dynamical system of (1.1) driven by degenerate noise, based on the notion of asymptotic strong Feller property in [4]. In this paper, we will consider the large deviation principle for (1.1) driven by small noise.

* Corresponding author.
E-mail address: xuekepu@cqu.edu.cn (X. Pu).

http://dx.doi.org/10.1016/j.aml.2015.02.014
0893-9659/

The large deviation principle (LDP) is one of the central areas in modern probability and statistics, and the LDP for stochastic partial differential equations has been considered in recent years. The framework was introduced by Varadham and developed by Azencott, Freidlin and Wenztell et al. for finite dimensional stochastic dynamical systems [5–8]. When the system is driven by additive noise, the LDP basically follows from the Varadham contraction principle. In the general case, Freidlin and Wenztell developed the discretization method for the LDP. Recently, it was found that the stochastic control and the weak convergence method can be used to show LDP by proving an equivalent Laplace principle (LP). This approach is mainly based on a variational representation formula for some functionals of infinite dimensional Brownian motion, and can avoid some exponential probability estimates. We would like to remark that the LDP for the one-dimensional stochastic Ginzburg–Landau equation with multiplicative noise was studied in [3] by this weak convergence approach. In this paper, we use the weak convergence method to study the LDP for the 3D cubic stochastic Ginzburg–Landau equation.

In Section 2, we give some preliminaries and state the main result of this paper, see Theorem 2.2. In Section 3, we prove this theorem by establishing the Laplace principle. Throughout this paper, we use $\langle u, v\rangle_0 = \int_D u\bar{v}dx$ to denote the L^2-inner product and $\|\cdot\|_X$ to denote the X-norm.

2. Preliminaries and main result

Let $(H_0, \|\cdot\|_0)$ and $(H, \|\cdot\|)$ be two Hilbert spaces. For a Polish space $\mathcal{E}$ and for $\varepsilon > 0$, we let $\mathcal{G}^\varepsilon : C([0, T]; H_0) \to \mathcal{E}$ to be a measurable map. Let $\mathcal{P}_2(H_0)$ denote the class of H_0-valued $\mathcal{F}_t$-predictable processes v that satisfy $\int_0^T \|v(s)\|_0^2 ds < \infty$ a.s. For any $M > 0$, we define

$$\mathcal{S}^M = \left\{ v \in L^2([0, T]; H_0) : \int_0^T \|v(s)\|_0^2 ds \le M \right\},$$

$$\mathcal{A}^M = \left\{ v \in \mathcal{P}_2(H_0) : v(\omega) \in S^M, P\text{-}a.s. \right\}.$$

When endowed with the weak topology in $\mathcal{P}_2(H_0)$, S^M is a compact Polish space [5].

We will let $A(u) = (\lambda + i\alpha)\Delta u + \gamma u - (\kappa + i\beta)|u|^2 u$.

Definition 2.1. A function $I : \mathcal{E} \to [0, \infty]$ is called a rate function if for each $M < \infty$, the level set $\{x \in \mathcal{E} : I(x) \le M\}$ is compact in $\mathcal{E}$.

Definition 2.2. Let I be a rate function on $\mathcal{E}$, a family X^ε of $\mathcal{E}$-valued random variables is said to satisfy the Laplace principle on $\mathcal{E}$ with rate function I if for all bounded continuous functions $h : \mathcal{E} \to \mathbb{R}$,

$$\lim_{\varepsilon \to 0} \varepsilon \log \mathbf{E} \left\{ \exp\left[-\frac{1}{\varepsilon} h(X^\varepsilon) \right] \right\} = - \inf_{x \in \mathcal{E}} \{h(x) + I(x)\}.$$

Assumption 2.1. Assume that there exists a measurable map $\mathcal{G}^0 : C([0, T]; H_0) \to \mathcal{E}$ such that

(i) For any $M > 0$, and a family $\{v^\varepsilon\} \subset \mathcal{A}^M$ such that $v^\varepsilon \to v^0$ in distribution as $\mathcal{S}^M$-valued random elements, then $\mathcal{G}^\varepsilon \left(W(\cdot) + \frac{1}{\sqrt{\varepsilon}} \int_0^\cdot v^\varepsilon(s) ds \right) \to \mathcal{G}^0 \left(\int_0^\cdot v^0(s) ds \right)$ in distribution as $\mathcal{E}$-valued random elements.

(ii) For every $M < \infty$, the set Γ_0^M is a compact subset in $\mathcal{E}$:

$$\Gamma_0^M := \left\{ \mathcal{G}^0 \left(\int_0^\cdot v^0(s) ds \right) : v^0 \in \mathcal{A}^M \right\}. \tag{2.1}$$

For each $f \in \mathcal{E}$, we let

$$I(f) := \inf_{\left\{ \substack{v^0 \in L^2([0,T];H_0),\\ f = \mathcal{G}^0(\int_0^\cdot v^0(s)ds)} \right\}} \left\{ \frac{1}{2} \int_0^T \|v^0(s)\|_0^2 ds \right\}, \tag{2.2}$$

with the convention that $\inf \emptyset = \infty$.

Theorem 2.1. *Let the family $\{\mathcal{G}^\varepsilon\}$ satisfy Assumption 2.1, then Laplace principle holds for $X^\varepsilon = \mathcal{G}^\varepsilon(W(\cdot))$.*

In our situation, we take $H_0 = l^2$, $H = H_0^1(D)$ and $\mathcal{E} = C([0, T]; H_0^1(D)) \cap L^2(0, T; H^2(D))$. Let $\{e_j = (0, \dots, 0, 1, 0, \dots) : j = 1, 2, \dots\}$ be an orthonormal basis of l^2. It was proved that under the above assumptions, there exists a unique strong solution $u_\varepsilon \in \mathcal{E}$ for (1.1) in [2]. Therefore, there exists a measurable functional $\mathcal{G}^\varepsilon : C([0, T]; H_0) \to \mathcal{E}$, such that $u_\varepsilon(t, \omega) = \mathcal{G}^\varepsilon(W(\cdot, \omega))(t)$ with $W(t) = \sum_{j=1}^\infty W_t^j e_j$. To verify the Laplace principle we need to consider

$$u^\varepsilon := \mathcal{G}^\varepsilon \left(W(\cdot) + \frac{1}{\sqrt{\varepsilon}} \int_0^\cdot v^\varepsilon(s) ds \right), \tag{2.3}$$