Applied Mathematics Letters 48 (2015) 91–101

Contents lists available at ScienceDirect

Applied Mathematics Letters

www.elsevier.com/locate/aml

How to improve the domain of parameters for Newton's method

J.A. Ezquerro*, M.A. Hernández-Verón

University of La Rioja, Department of Mathematics and Computation, C/ Luis de Ulloa s/n, 26004 Logroño, Spain

ARTICLE INFO

Article history:
Received 13 January 2015
Received in revised form 21 March 2015
Accepted 21 March 2015
Available online 11 April 2015

Keywords:
Newton's method
Semilocal convergence
Recurrence relations
Domain of parameters

ABSTRACT

We study the influence of a center Lipschitz condition for the first derivative of the operator involved when the solution of a nonlinear equation is approximated by Newton's method in Banach spaces. As a consequence, we see that the domain of parameters associated to the Newton–Kantorovich theorem is enlarged.

1. Introduction

In this study we are concerned with the problem of approximating a solution x^* of the equation $F(x) = 0$, where F is a nonlinear Fréchet differentiable operator defined on a nonempty open convex domain Ω of a Banach space X with values in a Banach space Y. It is usual to approximate x^* by using iterative methods. Among these, the best known and most used is Newton's method, whose algorithm is:

$$x_0 \text{ given in } \Omega, \quad x_{n+1} = x_n - [F'(x_n)]^{-1}F(x_n), \quad n = 0, 1, 2 \ldots . \tag{1}$$

In this paper, we focus our attention on the analysis of the semilocal convergence of sequence (1), which is based on demanding conditions to the initial approximation x_0, from certain conditions on the operator F, and provide the so-called domain of parameters corresponding to the conditions required to the initial approximation that guarantee the convergence of sequence (1) to the solution x^*.

The best known semilocal convergence result for Newton's method in Banach spaces is the Newton–Kantorovich theorem [1], which is given under the following general conditions:

* Corresponding author.
E-mail addresses: jezquer@unirioja.es (J.A. Ezquerro), mahernan@unirioja.es (M.A. Hernández-Verón).

http://dx.doi.org/10.1016/j.aml.2015.03.018

(A1) There exists $\Gamma_0 = [F'(x_0)]^{-1} \in \mathcal{L}(Y, X)$, for some $x_0 \in \Omega$, with $\|\Gamma_0\| \leq \beta$ and $\|\Gamma_0 F(x_0)\| \leq \eta$, where $\mathcal{L}(Y, X)$ is the set of bounded linear operators from Y to X.

(A2) There exists a constant $K \geq 0$ such that $\|F'(x) - F'(y)\| \leq K\|x - y\|$ for $x, y \in \Omega$.

Theorem 1 (*The Newton–Kantorovich Theorem*). *Let $F : \Omega \subseteq X \longrightarrow Y$ be a once continuously differentiable operator defined on a nonempty open convex domain Ω of a Banach space X with values in a Banach space Y. Suppose that conditions* (A1)–(A2) *are satisfied. If $K\beta\eta \leq \frac{1}{2}$ and $B(x_0, t^*) \subset \Omega$, where $t^* = \frac{1-\sqrt{1-2K\beta\eta}}{K\beta}$, then Newton's sequence, given by* (1)*, converges to a solution x^* of the equation $F(x) = 0$, starting at x_0, and $x_n, x^* \in \overline{B(x_0, t^*)}$, for all $n = 0, 1, 2, \ldots$.*

From condition (A2) of the Newton–Kantorovich theorem, we obtain the value of K once F is known. Moreover, every point $x \in \Omega$ such that the operator $[F'(x)]^{-1}$ exists with $\|[F'(x)]^{-1}\| \leq \beta$ and $\|[F'(x)]^{-1}F(x)\| \leq \eta$ has associated the pair $(K, \beta\eta)$ of the xy-plane, where $x = K$ and $y = \beta\eta$. In addition, if we consider the set

$$D = \left\{(K, \beta\eta) \in \mathbb{R}^2 : K\beta\eta \leq \frac{1}{2}\right\},$$

we can observe that every point x such that the pair associated $(K, \beta\eta)$ belongs to D can be chosen as starting point for Newton's method, so that the method converges to a solution x^* of the equation $F(x) = 0$ to start on it. As a consequence, we are interested in D is as big as possible, since this fact allows us to find a greater number of good starting points for Newton's method.

The set D associated to the Newton–Kantorovich theorem is called domain of parameters. We can draw D by choosing $x = K$ and $y = \beta\eta$ and coloring the region of the xy-plane whose points satisfy the condition $K\beta\eta \leq \frac{1}{2}$ (namely, $xy \leq \frac{1}{2}$) of the Newton–Kantorovich theorem (D is the black region of Figs. 1–2), so that the convergence of Newton's method is guaranteed from this condition. Note that condition (A1), required to the initial approximation x_0, define the parameters β and η, and condition (A2), required to the operator F, define the fixed parameter K.

The main aim of this work is to improve the domain of parameters D under conditions (A1)–(A2). To modify the set D, we can use different strategies that change condition (A2), as we can see in [2–5], but these modifications do not increase the size of D. Argyros gives in [2] a procedure to increase the size of D by noticing that, as a consequence of condition (A2), once $x_0 \in \Omega$ is fixed, the condition

$$\|F'(x) - F'(x_0)\| \leq K_0\|x - x_0\|, \quad x \in \Omega, \tag{2}$$

is satisfied with $K_0 \leq K$. In addition, Argyros, under conditions (A1) and (A2) and taking into account (2), proves the semilocal convergence of (1) by using the majorant principle of Kantorovich [6]. The optimum case of Argyros occurs when $K\beta\eta \leq \frac{1}{1+\mu}$, where $\mu = \frac{K_0}{K} \in (0, 1]$. We can see then that the domain of parameters associated is

$$D_1 = \left\{(K, \beta\eta) \in \mathbb{R}^2 : \; h = K\beta\eta \leq \frac{1}{1+\mu}, \; \mu = \frac{K_0}{K} \in (0, 1]\right\},$$

which is bigger than that of the Newton–Kantorovich theorem, since $D \subset D_1$ for $\mu < 1$, as we can see in Fig. 1. In addition, from Fig. 1, we can guess that the smaller the quantity $\mu = \frac{K_0}{K} \in (0, 1]$ is, the bigger the domain of parameters is: yellow for $\mu = 0.75$, red for $\mu = 0.5$, green for $\mu = 0.25$ and orange for $\mu = 0.1$. Besides, if $\mu \to 1$, the domain of parameters associated to the semilocal convergence result given by Argyros in [2] tends to be that obtained by the Newton–Kantorovich theorem (black region).

In this paper, we prove the semilocal convergence of Newton's method by using a technique based on recurrence relations which takes into account the previous idea introduced by Argyros. As a result, the sizes of D and D_1 are improved.

Throughout the paper we denote $\overline{B(x, \varrho)} = \{y \in X; \|y - x\| \leq \varrho\}$ and $B(x, \varrho) = \{y \in X; \|y - x\| < \varrho\}$.