Applied Mathematics Letters 47 (2015) 35–42

Contents lists available at ScienceDirect

Applied Mathematics Letters

www.elsevier.com/locate/aml

A system of coupled partial differential equations exhibiting both elevation and depression rogue wave modes

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ARTICLE INFO

Article history:
Received 28 January 2015
Received in revised form 25 February 2015
Accepted 26 February 2015
Available online 7 March 2015

Keywords:
Breathers
Algebraic solitons
Rogue waves

ABSTRACT

Analytical solutions are obtained for a coupled system of partial differential equations involving hyperbolic differential operators. Oscillatory states are calculated by the Hirota bilinear transformation. Algebraically localized modes are derived by taking a Taylor expansion. Physically these equations will model the dynamics of water waves, where the dependent variable (typically the displacement of the free surface) can exhibit a sudden deviation from an otherwise tranquil background. Such modes are termed 'rogue waves' and are associated with 'extreme and rare events in physics'. Furthermore, elevations, depressions and 'four-petal' rogue waves can all be obtained by modifying the input parameters.

1. Introduction

Rogue waves are unexpectedly large displacements from an equilibrium position or an otherwise tranquil background [1,2]. Even though such dangerous waves have been known to the maritime community for nearly a century, large scale theoretical studies in hydrodynamics began only recently [1]. With the observation of rogue wave modes in optical fibers as waveguides, studies of such large amplitude motions have been pursued across a broad spectrum of physical disciplines, under the general category of 'extreme and rare events in physics' [2].

The widely used model for rogue waves is the nonlinear Schrödinger (NLS) equation (γ = a real parameter, $*$ = complex conjugate),

$$i\Psi_t + \Psi_{xx} + \gamma \Psi^2 \Psi^* = 0, \tag{1}$$

where the complex valued, slowly varying wave envelope Ψ evolves under the influence of quadratic dispersion and cubic nonlinearity (t, x being slow time and group velocity coordinate in fluid mechanics respectively) [3].

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http://dx.doi.org/10.1016/j.aml.2015.02.021
0893-9659/

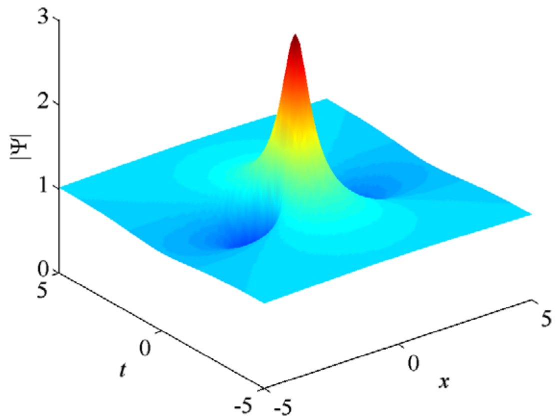

Fig. 1. (Color online) Peregrine soliton [Eq. (2)] for the nonlinear Schrödinger equation [Eq. (1)] with $\gamma = r = 1$.

Rogue wave modes (Peregrine solitons) are analytical solutions algebraically localized in x and t [4]:

$$\Psi = r\exp\left(i\gamma r^2 t\right)\left\{1 - \frac{2\left(1 + 2i\gamma r^2 t\right)}{\gamma r^2\left(x^2 + 2\gamma r^2 t^2 + \frac{1}{2\gamma r^2}\right)}\right\}, \quad r \text{ real}, \tag{2}$$

and are only nonsingular for $\gamma > 0$. The main displacement occurs near $x = t = 0$ as an elevation above the background plane (or continuous) wave $\Psi = r\exp(i\gamma r^2 t)$. Eq. (1) conserves the intensity $\int_{-\infty}^{\infty} |\Psi|^2\, dx$ for localized boundary conditions, and hence there will be accompanying depressions nearby (Fig. 1).

For special coupled NLS equations (commonly known as the Manakov system) with two components (Ψ and Φ),

$$i\Psi_t + \Psi_{xx} + \gamma(\Psi\Psi^* + \Phi\Phi^*)\Psi = 0, \qquad i\Phi_t + \Phi_{xx} + \gamma(\Psi\Psi^* + \Phi\Phi^*)\Phi = 0,$$

nonsingular algebraically localized modes can also occur for $\gamma < 0$, in sharp contrast with the single component case Eq. (1) [5]. The main displacement is then a depression below the mean level in the center of the x, t plane. The character of the rogue wave mode (elevation or depression) thus appears to depend critically on the parameters of the partial differential equations.

Other than the NLS systems, many other evolution equations exhibit rogue wave modes, e.g. the Hirota equation [6], the Kadomtsev–Petviashvili equation [7], the long wave–short wave resonance model [8], and systems displaying $\mathcal{PT}$-symmetry [9]. The goal of the present work is to propose still another system of partial differential equations (PDEs) which possesses rogue wave modes. The novel characters include:

- Formulations for breathers (pulsating modes) and rogue waves are given.
- The transition in wave profiles among 'elevations', 'depressions' and 'four-petal configurations' results from variation in one single parameter in the solution of the PDEs, and not the PDEs themselves. Physically this parameter is the wave number of the carrier wave packet. In other words, for a fixed system of PDEs, different families of wave profiles can be observed by changing the input wavelength.

For the widely studied NLS equation of one single complex valued component, rogue waves are possible only if dispersion and nonlinearity are of the same sign [1,2]. The situation is more intriguing for two or more components. Similarities and differences between the proposed system and the known ones will be highlighted in the discussion on wave profiles (Section 4).

• The appropriate range of this parameter can also be predicted precisely from an analysis of modulation instability (i.e. linear stability of the plane wave).