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Global convergence of the second order Ricker equation

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A B S T R A C T

We present a short analytic proof of a 1976 conjecture on the global dynamics of the equation $x_n = x_{n-1}e^{x-x_{n-2}}$, where $x \in (0,1)$ and $x_0, x_1 \in (0,+\infty)$. The proof is based on considering the parameter x in the previous equation as a complex variable. This transforms the problem in studying the asymptotic behaviour of a sequence of analytic functions.

1. Introduction

The difference-delayed Ricker equation,

$$x_n = x_{n-1}e^{x-x_{n-2}}, \qquad x_0, x_1 \in (0,+\infty), \tag{1}$$

where $x > 0$, arises as the natural generalization of the celebrated Ricker population model [1], $x_n = x_{n-1}e^{x-x_{n-1}}$, when there are explicit time lags in the density dependent regulatory mechanisms of the population.

In 1976, Levin and May [2] conjectured that the sequence in (1) converges to the equilibrium x, whenever $x \in (0,1)$. Recently, Bartha, Garab and Krisztin provided a computer aided proof for this conjecture [3]. Indeed, they showed that the sequence in (1) converges for the critical parameter value $x = 1$ as well. Interestingly, their proof is computer-aided for $x \in (0.5,1]$, but it is fully analytic for $x \in (0,0.5]$. Therefore, they gave an analytic proof partially solving the conjecture. However, as they noted, there were other previous analytic proofs partially solving the conjecture in a bigger subinterval of $(0,1)$; e.g. [4,5]. The best of these results was obtained by Tkachenko and Trofimchuk in [5], proving it for $x \in (0,0.875)$; see [3] for more details and other contributions related to this problem.

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In this letter, we give an analytic proof of the conjecture. Our approach is new: we treat the parameter x in (1) as a complex variable, studying the asymptotic behaviour of the sequence of analytic functions thus obtained. The proof is based on a combination of classical complex analysis theory and the study of the asymptotic behaviour of a linear difference equation. The organization of the paper is as follows. In Section 2 we prove some general properties of the sequence in (1) which are needed for later sections. In particular, we show that if the sequence converges, then it does it rapidly (in the sense that the series of the absolute differences between x_n and its limit converges). In Section 3 we focus on the dynamics of a second order linear difference equation, providing asymptotic bounds for its solutions (Theorem 2). This result is used in Section 4 to get a priori bounds for the derivatives of the sequence of analytic functions commented above, but we consider Theorem 2 of independent interest. Finally, also in Section 4 we analytically prove the conjecture showing, in addition, that the convergence is uniform on compact sets of $(0, 1)$.

2. Preliminaries

In the following result we provide a new short proof for the conjecture in the case $x \in (0, 1/2]$ and we prove that, if x_n converges, then it does it rapidly.

Theorem 1. *Let $x, x_0, x_1 \in (0, +\infty)$ and define $x_n = x_{n-1}e^{x-x_{n-2}}$ for $n \geq 2$. Then, $0 < x_n \leq e^{2x-1}$ for $n \geq 2$ and $\liminf_{n\to\infty} x_n \leq x \leq \limsup_{n\to\infty} x_n$. Moreover,*

(a) *if $x \in (0, 1/2]$, then $\lim_{n\to\infty} x_n = x$.*
(b) *if $x \in (0, 1)$ and $\lim_{n\to\infty} x_n = x$, then $\sum_{n=0}^{\infty} |x_n - x| < \infty$.*

Proof. Notice that

$$0 < x_n = x_{n-1}e^{x-x_{n-2}} = x_{n-2}e^{x-x_{n-2}}e^{x-x_{n-3}} \leq e^{x-1}e^{x} = e^{2x-1}.$$

Let μ and ν be the upper and lower limits of x_n respectively. We have that $\nu \leq x \leq \mu$. Otherwise, x_n would be an eventually decreasing sequence in (x, ∞) or an eventually increasing sequence in $(0, x)$, thus it would converge to the equilibrium x and the conclusion would follow.

Assume now that $x \in (0, 1/2]$, so $x_n \leq 1$ for $n \geq 2$. Let $x_{\sigma(n)}$ be a subsequence of x_n converging to μ. The sequence $(x_{\sigma(n)}, x_{\sigma(n)-1}, x_{\sigma(n)-2}, x_{\sigma(n)-3})$ is bounded in $\mathbb{R}^4$, thus it has a subsequence converging to a point $(\mu, \gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}^4$, which satisfies $\mu = \gamma_1 e^{x-\gamma_2}$ (thus $\gamma_2 \leq x$) and $\mu = \gamma_2 e^{x-\gamma_2}e^{x-\gamma_3}$. As $h(t) = te^{-t}$ is an increasing function in $(-\infty, 1]$, we have

$$\mu = \gamma_2 e^{x-\gamma_2}e^{x-\gamma_3} \leq xe^{x-x}e^{x-\nu} \leq \mu e^{x-\mu}e^{x-\nu},$$

and therefore $\mu \leq xe^{x-\nu}$ and $2x - \mu - \nu \geq 0$. Analogously, by considering a subsequence $x_{\rho(n)} \to \nu$, we have $2x - \mu - \nu \leq 0$. Now, $\mu \leq xe^{x-\nu} = xe^{x-(2x-\mu)}$, that is, $\mu e^{-\mu} \leq xe^{-x}$, which implies $\mu = x$, $\nu = 2x - \mu = x$.

Finally, assume $x \in (0, 1)$ and $\lim_{n\to\infty} x_n = x$. It can be easily seen that $x_n = x_1 e^{\sum_{k=0}^{n-2}(x-x_k)}$, so $\sum_{k=0}^{\infty}(x - x_k) = \log(x/x_1)$. To verify that the series converges absolutely, we consider the following sets:

$$M_1 := \{n \in \mathbb{N} : x_n \leq \min\{x_{n-1}, x_{n+1}\}\},$$
$$M_2 := \{n \in \mathbb{N} : x_n \geq \max\{x_{n-1}, x_{n+1}\}\},$$

and $M := M_1 \cup M_2$. If M_1 or M_2 are finite sets, then the sequence $x - x_k$ is eventually monotonic and then the series converges absolutely. In other case, let $\sigma : \mathbb{N} \to M$ be the increasing bijection. Notice that