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Uniqueness of weak solutions for a pseudo-parabolic equation modeling two phase flow in porous media

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ABSTRACT

In this paper, we prove the uniqueness of weak solutions for a pseudo-parabolic equation modeling two-phase flow in a porous medium, where dynamic effects are included in the capillary pressure. We transform the equation into an equivalent system, and then prove the uniqueness of weak solutions to the system which leads to the uniqueness of weak solutions for the original model.

1. Introduction

We consider a two-phase flow model in a porous medium which includes dynamic effects in the capillary pressure. If determined under equilibrium condition, the difference of the pressures in the two phases (wetting and non-wetting) is a decreasing function in terms of the (say) wetting phase saturation u

$$p_n - p_w = p_c(u). \tag{1}$$

Existence and uniqueness as such models are analyzed in [1,2]. However, the equilibrium assumption does not hold in several real life applications, like paper drying. Further, experimental evidence the non-validity of the equilibrium assumption are provided e.g. in [3–5]). In this case, dynamic effects have to be included. An example in this sense is the model proposed in [6]:

$$p_n - p_w = p_c(u) - \tau \partial_t u. \tag{2}$$

Here $\tau > 0$ is a parameter accounting for the dynamic effects.

Inspired by this, we consider here a simplified model obtained from mass conservation and the Darcy law for each phase (see [7,8]), assuming that the medium is fully saturated by the two phases. Generally, this leads to a system of two equations, a parabolic one for the wetting phase saturation, and an elliptic one for the total flux. Here we assume the latter known, and focus on the mass balance for the wetting phase. With the phase pressure difference in (2), the model reads (see [9,10])

$$\partial_t u + \nabla \cdot \mathbf{F}(u) + \nabla \cdot (H(u)\nabla(p_c(u) - \tau \partial_t u)) = 0. \tag{3}$$

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It is defined in a bounded and connected domain Ω in $\mathbb{R}^d$ ($d = 1, 2, 3$) with a given time interval $(0, T]$. Further, by $\bar{\Omega}$ we mean the closure of Ω, and by $\partial\Omega$ its boundary. In the above, $\mathbf{F}$ and H denote the water fractional flow function and the capillary induced diffusion function, and p_c is the equilibrium capillary pressure (see (1) and (2)). These are non-linear functions defined for the physically relevant interval $u \in [0, 1]$. For the mathematical analysis, we extend H, p_c and $\mathbf{F}$ continuously to the entire $\mathbb{R}$. Throughout this paper, we assume

A1: $H: \mathbb{R} \longrightarrow \mathbb{R}$ is Lipschitz continuous, and a h_0 exists such that

$$0 < h_0 \le H(u) \quad \text{if } 0 < u < 1, \quad \text{and} \quad H(u) = h_0 \text{ otherwise.} \tag{4}$$

A2: $p_c \in C^1(\mathbb{R})$ is a decreasing function, and m_p, M_p exist such that $0 < m_p \le |p_c'(u)| \le M_p < +\infty$, for all $u \in \mathbb{R}$.

A3: $\mathbf{F}: \mathbb{R} \longrightarrow \mathbb{R}^d$ is Lipschitz continuous.

A4: The functions $\mathbf{F}$ and H are bounded, $|\mathbf{F}(u)| + |H(u)| \le M < +\infty$, for all $u \in \mathbb{R}$.

To complete the model, the initial and boundary conditions are given

$$u(\cdot, 0) = u^0, \quad \text{and} \quad u = 0, \quad \text{at } \partial\Omega. \tag{5}$$

For the initial data, we assume

A5: $u^0 \in C^{0,\alpha}(\bar{\Omega})$ for some $\alpha \in (0, 1]$, $u^0 = 0$ at $\partial\Omega$ and $u^0 \in [0, 1]$ in Ω.

Further, the domain Ω has a smooth boundary:

A6: Ω is a $C^{1,\alpha}$ domain.

The motivation for (3) is the two phase flow in porous media. Generally, such models are not only non-linear, but may also degenerate. Whenever one of the phase is not present. More precise, functional relationships between quantities: the permeability, pressure, and saturation lead to cases when, e.g. $H(u) = 0$ for $u = 0$ or $u = 1$ (see [7,11,12]). Then the highest order term on (3) is vanishing, and the equation is not parabolic anymore. This makes the analysis and numerical simulation of such problems complicated. A usual technique to overcome the difficulties related to degeneracy is regularization. For example, one can approximate non-linearities by regular permeabilities, which are bounded away from 0 and ∞. The uniqueness result obtained here works in the regularized cases, and was still an open issue even for regular cases.

In this paper we prove the uniqueness of a weak solution to (3) with the given initial and boundary conditions. This solution solves

Problem P. Find $u \in W^{1,2}(0, T; W_0^{1,2}(\Omega))$, such that $u(\cdot, 0) = u^0$, $\nabla\partial_t u \in L^2(0, T; L^2(\Omega)^d)$ and

$$\int_0^T \int_\Omega \partial_t u\phi dxdt - \int_0^T \int_\Omega \mathbf{F}(u) \cdot \nabla\phi dxdt - \int_0^T \int_\Omega H(u)\nabla p_c(u) \cdot \nabla\phi dxdt + \tau \int_0^T \int_\Omega H(u)\nabla\partial_t u \cdot \nabla\phi dxdt = 0, \tag{6}$$

for any $\phi \in L^2(0, T; W_0^{1,2}(\Omega))$.

Note the non-linearity appearing in the highest order term, $\nabla \cdot (H(u)\nabla\partial_t u)$. For the linear case, when only $\Delta\partial_t u$ is appearing, existence and uniqueness results are obtained in [13]. Also we refer to [9,10,14] for the existence of weak solutions to the nonlinear Problem P. However, the uniqueness in the case considered here is still an open issue. To overcome the difficulty related to the non-linearity appearing in the highest order term, we use an additional unknown p to represent the phase pressure difference (see [15,16]). Closest to our work are the recent results in [15]. There, the uniqueness is obtained for a similar, even degenerate model, but in the absence of convective terms, and more important, when the non-linearities appear only under all derivatives e.g. $\partial_t\Delta\psi(u)$. Apart from some particular cases (i.e. $H \equiv$ *constant*), the model considered here involves the term $\nabla \cdot (H(u)\nabla\partial_t u)$, which cannot be transformed to the form in [15]. Furthermore, the uniqueness in [15] is proved for an alternative formulation when (3) is written as a system. In the non-degenerate case, the equivalence between (3) and its reformulation as a system (like (7)–(8)) is proved in [16], while in the degenerate case, it is still open. Thus, uniqueness results in [15], whenever the model considered here matches the frame work there, gives also uniqueness for (3). However, this is only for particular cases, as mentioned. Besides, in this paper, we provide an alternative uniqueness proof, based on Green functions.

To avoid the confusion between u given by Problem P and the solution pair of the following system, we denote the saturation latter by s. With this, (3) can be rewritten formally as the system

$$\partial_t s + \nabla \cdot \mathbf{F}(s) + \nabla \cdot (H(s)\nabla p) = 0, \tag{7}$$

$$p = p_c(s) - \tau\partial_t s. \tag{8}$$

For the sake of presentation we finally assume

A7: $p_c(0) = 0$.

Remark 1. A7 can be avoided: if $p_c(0) \ne 0$, one defines $p = p_c(s) - p_c(0) - \tau\partial_t s$ in (8).