Applied Mathematics Letters 46 (2015) 50–56

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

The closed-form particular solutions for Laplace and biharmonic operators using a Gaussian function

A.R. Lamichhane, C.S. Chen *

Department of Mathematics, University of Southern Mississippi, Hattiesburg, MS, USA

ARTICLE INFO

Article history:
Received 3 December 2014
Received in revised form 7 February 2015
Accepted 8 February 2015
Available online 18 February 2015

Keywords:
Radial basis functions
Particular solution
Meshless method
Method of particular solution
Localized method of approximate particular solutions
Gaussian RBF

ABSTRACT

Particular solutions play a critical role in solving inhomogeneous problems using boundary methods such as boundary element methods or boundary meshless methods. In this short article, we derive the closed-form particular solutions for the Laplace and biharmonic operators using the Gaussian radial basis function. The derived particular solutions are implemented numerically to solve boundary value problems using the method of particular solutions and the localized method of approximate particular solutions. Two examples in 2D and 3D are given to show the effectiveness of the derived particular solutions.

1. Introduction

In the literature of the boundary element method, the Dual Reciprocity Method (DRM) has been developed to avoid the domain integration. To successfully implement the DRM [1], a closed-form particular solution is essential. A great deal of effort has been devoted to derive the closed-form particular solution using radial basis functions (RBFs) [2–4]. In recent years, various types of meshless methods using radial basis functions (RBFs) have been developed to efficiently solve partial differential equations. Among them, the method of particular solutions (MPS) [5,6], the localized method of approximate particular solutions (LMAPS) [7], and MFS-MPS [8] are the meshless collocation methods which require the use of closed-form particular solutions as the basis functions in the solution process. Hence, the closed-form particular solutions become the core of these particular solutions based meshless methods. The importance of the closed-form particular solutions to the above mentioned RBF-based meshless methods is analogous to the fundamental solutions to the boundary element methods.

The Gaussian RBF has been widely used in the area of neural networks [9]. However, the Gaussian RBF is rarely being used for solving partial differential equations [10]. In the RBF literature, the closed-form particular solutions using MQ, polyharmonic splines, and compactly supported radial basis functions have been derived for the above mentioned particular solution based meshless collocation methods [2,11,3,4]. In the past, the closed-form particular solution for the Laplace operator using the Gaussian as a basis function was not available due to the difficulty of the integration involving the Gaussian RBF. In recent years, the symbolic computational tools such as MATHEMATICA and MAPLE have made the difficult integration task possible. Furthermore, the evaluations of special functions such as Bessel functions, error function and exponential integral functions are available as library functions or built-in functions in many computational platforms such as MATLAB, FORTRAN and C++. These special functions which often involve infinite series have been considered as the closed-form functions

* Corresponding author.
E-mail address: cschen.math@gmail.com (C.S. Chen).

http://dx.doi.org/10.1016/j.aml.2015.02.004
0893-9659/

and can be evaluated efficiently and accurately. Motivated by the availability of these new computational tools, we re-investigate the possibility of deriving the particular solutions for various differential operators using the Gaussian RBF.

The main goal of this study is to focus on the derivation of the closed-form particular solutions using the Gaussian RBF. For the numerical implementation using the MPS and LMAPS, we refer readers to Refs. [5–7] for further details. To validate the particular solutions we derive in this short article, we apply the MPS for a 2D problem and LMAPS for a 3D problem. Note that the MPS is a global RBF meshless method and LMAPS is a localized meshless method which is capable of handling a large number of collocation points.

The structure of the paper is as follows. In Section 2, we derive the particular solutions of Laplace and biharmonic differential operators in 2D and 3D using the Gaussian RBF. In Section 3, we perform the test on two examples to demonstrate the effectiveness of the derived particular solutions. Some concluding remarks are placed in the last section.

2. Particular solution

In this section, we will derive the solution to the inhomogeneous Laplace and biharmonic operators with a Gaussian RBF right hand side.

2.1. Particular solution for Laplace operator

Theorem 1. *Let $\phi(r) = \exp(-cr^2)$, $c > 0$, and $\Delta\Phi(r) = \phi(r)$ in 2D. Then,*

$$\Phi(r) = \begin{cases} \frac{1}{4c}\,\mathrm{Ei}(cr^2) + \frac{1}{2c}\log(r), & r \neq 0, \\ \frac{-1}{4c}(\gamma + \log(c)), & r = 0, \end{cases} \tag{1}$$

where

$$\mathrm{Ei}(x) = \int_x^\infty \frac{e^{-u}}{u}\,du, \tag{2}$$

and $\gamma \simeq 0.5772156649015328$ is the Euler–Mascheroni constant [12]. *Note that* $\mathrm{Ei}(x)$ *is the special function known as the exponential integral function* [12].

Proof. Suppose

$$\Delta\Phi = \exp(-cr^2). \tag{3}$$

In polar co-ordinates in 2D, for radial invariant functions we have

$$\Delta = \frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right). \tag{4}$$

By direct integration on both sides of (3), we have

$$\begin{aligned} r\frac{d\Phi}{dr} &= \int r\exp(-cr^2)dr \\ &= -\frac{1}{2c}\exp(-cr^2) + C_0. \end{aligned} \tag{5}$$

It follows that

$$\Phi(r) = \frac{1}{4c}\,\mathrm{Ei}(cr^2) + C_0\log(r) + C_1, \tag{6}$$

where C_0 and C_1 are integration constants. Note that [12]

$$\mathrm{Ei}(cr^2) = -\gamma - \log(cr^2) + cr^2 + \frac{c^2r^4}{4} + O(r^5), \tag{7}$$

which contains a singular term at $r = 0$. By choosing the integration constant $C_0 = 1/2c$, we can de-singularize $\Phi(r)$ in (6). Another integration constant C_1 in (6) can be chosen arbitrarily. For convenience, we set $C_1 = 0$. Hence, $\Phi(r)$ in (1) is proved.

Furthermore,

$$\begin{aligned} \lim_{r\to 0}\Phi(r) &= \lim_{r\to 0}\left(\frac{1}{4c}\,\mathrm{Ei}(cr^2) + \frac{1}{2c}\log(r)\right) \\ &= -\frac{1}{4c}(\gamma + \log(c)). \quad \blacksquare \end{aligned}$$