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A finite difference scheme for nonlinear ultra-parabolic equations

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ABSTRACT

In this paper, our aim is to study a numerical method for an ultraparabolic equation with nonlinear source function. Mathematically, the bibliography on initial–boundary value problems for ultraparabolic equations is not extensive although the problems have many applications related to option pricing, multi-parameter Brownian motion, population dynamics and so forth. In this work, we present the approximate solution by virtue of finite difference scheme and Fourier series. For the nonlinear case, we use an iterative scheme by linear approximation to get the approximate solution and obtain error estimates. A numerical example is given to justify the theoretical analysis.

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1. Introduction

Let H be a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$. In this paper, we consider the problem of finding $u : [0, T] \times [0, T] \to H$ satisfies the following ultraparabolic equation

$$\begin{cases} \dfrac{\partial}{\partial t} u(t,s) + \dfrac{\partial}{\partial s} u(t,s) + Au(t,s) = f(u(t,s), t, s), & (t,s) \in (0,T) \times (0,T), \\ u(0,s) = \varphi(s), \quad s \in [0,T], \\ u(t,0) = \psi(t), \quad t \in [0,T]. \end{cases} \tag{1.1}$$

where $A : D(A) \subset H \to H$ is a positive-definite, self-adjoint operator with compact inverse on H and φ, ψ are known smooth functions satisfying $\varphi(0) = \psi(0)$ for compatibility at $(t, s) = (0, 0)$ and f is a source function which is defined later.

The problem (1.1) involving multi-dimensional time variables is called the initial–boundary value problem for ultraparabolic equation. The ultraparabolic equation has many applications in mathematical finance (e.g. [1]), physics (such as multi-parameter Brownian motion [2]) and biological models. Among many applications, the ultraparabolic equation arises as a mathematical model of population dynamics. The study of ultraparabolic equation for population dynamics can be found in some papers such as [3,4]. In particular, Kozhanov [4] studied the existence and uniqueness of regular solutions and its

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properties for an ultraparabolic model equation in the form of

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial s} - \Delta u + h(x,t,s)\,u + Au = f(x,t,s),$$

where Δ is Laplace operator, A is a nonlocal linear operator. In the same work, Deng and Hallam in [3] considered the age structured population problem in the form of

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial s} - \nabla \cdot (k\nabla u - qu) = -\mu u,$$

associated with non-locally integro-type initial-bounded conditions.

The ultraparabolic equation is also studied in many other aspects. In the study of inverse problems, Lorenzi [5] studied the well-posedness of a class of forward problems for ultraparabolic partial integrodifferential equations. Very recently, Zouyed and Rebbani [6] proposed the modified quasi-boundary value method to regularize a final value problem for (1.1). For other studies regarding the properties of solutions of abstract ultraparabolic equations, we can find many papers and some of them are [7–12].

Numerical methods for ultraparabolic equation are studied long time ago, for example [13–16]. Since 1996, Akrivis, Crouzeix and Thomée [14] investigated a backward Euler scheme and second-order box-type finite difference procedure to numerically approximate the solution to the Dirichlet problem for the ultraparabolic equation (1.1) in two different time intervals with the Laplace operator $A = -\Delta$ and the source function $f \equiv 0$. As opposed to the method-of-lines approach developed in [14], the well-posedness of the ultraparabolic problem, utilizing the directional derivative and demonstrated in Tersenov et al. [13], yields a method-of-characteristics numerical scheme. Recently, Ashyralyev and Yilmaz [15] constructed the first and second order difference schemes to approximate the problem (1.1) in case that $f \equiv 0$ for strongly positive operator and obtained some fundamental stability results. On the other hand, Marcozzi et al. [16] developed an adaptive method-of-lines extrapolation discontinuous Galerkin method for an ultraparabolic model equation given by $\frac{\partial u}{\partial t} + a(x)\frac{\partial u}{\partial s} - \Delta u = f(x,t,s)$, with a certain application to the price of an Asian call option. However, most of papers for numerical methods aim to study linear cases. Equivalently, numerical methods for nonlinear equations are still limited. Motivated by this reason, in this paper, we develop a finite difference method for a nonlinear ultraparabolic equation. Our method comes from the idea of Ashyralyev et al. [15,17], but it is different to their method. We remark that the numerical approach by Ashyralyev et al. [15] is computationally expensive, especially using finite difference scheme in space. Normally, the matrices generated by space-discretization are very big if the spatial dimension is high. Therefore, in this paper we shall study the model problem (1.1) in the numerical angle for the smooth solution by a different approach in space. Theoretically, paying attention to the idea of finite difference scheme in time, studied in e.g. [15,17,18] by Ashyralyev et al., and conveying a fundamental result in operator theory, we construct an approximate solution for problem (1.1) in terms of Fourier series.

The rest of the paper is organized as follows. In Section 2, a finite difference scheme is proposed. Furthermore, stability and convergence of the proposed scheme is established. Finally, a numerical example is implemented in Section 3 to verify the effectiveness of the method.

2. Finite difference scheme for nonlinear ultra parabolic : stability analysis

In this section, we consider a numerical method for Problem (1.1) in the case that f satisfies the global Lipschitz condition

$$\|f(u,t,s) - f(v,t,s)\| \le K\|u - v\|, \tag{2.2}$$

where K is a positive number independent of u, v, t, s. By simple calculation analogous to the steps in linear nonhomogeneous case, we get the discrete solution, then use linear approximation to get the explicit form of the approximate solution.

From now on, suppose that $A : D(A) \subset H \to H$ is a positive-definite, self-adjoint operator with a compact inverse in H. As a consequence, A admits an orthonormal eigenbasis $\{\phi_n\}_{n\ge 1}$ in H, associated with the eigenvalues such that

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \lim_{n\to\infty} \lambda_n = \infty.$$

With $G_n(t,s) = \exp\left(\frac{\lambda_n}{2}(t+s)\right)$, by a simple computation, the problem (1.1) is transformed into the following problem.

$$\begin{cases} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right)\left(\left\langle u(t,s), \phi_n\right\rangle G_n(t,s)\right) = \langle f(u(t,s),t,s), \phi_n\rangle G_n(t,s), \\ \langle u(0,s), \phi_n\rangle G_n(t,s) = \langle \varphi(s), \phi_n\rangle G_n(t,s), \\ \left\langle u(t,0), \phi_n\right\rangle G_n(t,s) = \langle \psi(t), \phi_n\rangle G_n(t,s). \end{cases} \tag{2.3}$$

For the numerical solution of this problem by finite difference scheme as introduced in the above section, a uniform grid of mesh-points $(t,s) = (t_k, s_m)$ is used. Here $t_k = k\omega$ and $s_m = m\omega$, where k and m are integers and ω the equivalent mesh-width in time t and s. We shall seek a discrete solution $u^{k,m} = u(t_k, s_m)$ determined by an equation obtained by replacing

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