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Systems of critical elliptic equations involving Hardy-type terms and large ranges of parameters

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ABSTRACT

In this paper, we study systems of elliptic equations involving critical nonlinearities and different Hardy-type terms. By variational methods, the existence of minimizers to Rayleigh quotients and ground state solutions to the systems is proved for large ranges of parameters.

1. Introduction and main results

In this paper, we are concerned with the following system of elliptic equations:

$$\begin{cases} -\Delta u - \lambda_1(x)\dfrac{u}{|x|^2} = \nu_1\big(\nu_1|u|^q + \nu_2|v|^q\big)^{\frac{2^*}{q}-1}|u|^{q-2}u, & \text{in } \mathbb{R}^N, \\ -\Delta v - \lambda_2(x)\dfrac{v}{|x|^2} = \nu_2\big(\nu_1|u|^q + \nu_2|v|^q\big)^{\frac{2^*}{q}-1}|v|^{q-2}v, & \text{in } \mathbb{R}^N, \\ (u, v) \in D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $2^* := \frac{2N}{N-2}$ is the critical Sobolev exponent, $D := D^{1,2}(\mathbb{R}^N)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to $\|u\| := (\int_{\mathbb{R}^N} |\nabla u|^2)^{1/2}$ and the parameters satisfy:

(H_1) $N \geq 3,\ \ \nu_1 > 0,\ \ \nu_2 > 0,\ \ 1 < q \leq 2^*$.

(H_2) $\lambda_i(x) \in C(\mathbb{R}^N)$ and there exist the limits $\lambda_i(\infty) := \lim\limits_{|x|\to\infty} \lambda_i(x)$ such that

$$\lambda_i(0) = \lambda_i(\infty) = \inf_{x\in\mathbb{R}^N} \lambda_i(x), \qquad \bar{\lambda}_i := \sup_{x\in\mathbb{R}^N} \lambda_i(x) < \Lambda_N := \left(\frac{N-2}{2}\right)^2, \quad i = 1, 2.$$

(H_3) $0 < \lambda_1(0) < \bar{\lambda}_1,\ 0 \leq \lambda_2(0) \leq \lambda_1(0)$.

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The energy functional of (1.1) is defined on $\mathbb{D} := D \times D$ by

$$I(u, v) = \frac{1}{2}\int_{\mathbb{R}^N}\Big(|\nabla u|^2 + |\nabla v|^2 - \frac{\lambda_1 u^2 + \lambda_2 v^2}{|x|^2}\Big)dx - \frac{1}{2^*}\int_{\mathbb{R}^N}\big(\nu_1|u|^q + \nu_2|v|^q\big)^{\frac{2^*}{q}}dx.$$

Then $I \in C^1(\mathbb{D}, \mathbb{R})$ and the duality product between $\mathbb{D}$ and its dual space $\mathbb{D}^{-1}$ is defined as $\langle I'(u, v), (\varphi, \psi)\rangle$, where (u, v), $(\phi, \psi) \in \mathbb{D}$, $I'(u, v)$ is the Fréchet derivative. $(u, v) \in \mathbb{D}\backslash\{(0, 0)\}$ is a solution of (1.1) if $\langle I'(u, v), (\phi, \psi)\rangle = 0,\ \forall(\phi, \psi) \in \mathbb{D}$.

Note that (1.1) is related to the Hardy inequality [1]:

$$\int_{\mathbb{R}^N}\frac{u^2}{|x|^2}dx \le \frac{1}{\Lambda_N}\int_{\mathbb{R}^N}|\nabla u|^2\,dx, \quad \forall\, u \in C_0^\infty(\mathbb{R}^N). \tag{1.2}$$

By (1.2) we can define the Sobolev-type constant for all $\lambda \in [0, \Lambda_N)$:

$$S(\lambda) := \inf_{u\in D\backslash\{0\}}\frac{Q_\lambda(u)}{\big(\int_{\mathbb{R}^N}|u|^{2^*}dx\big)^{\frac{2}{2^*}}}, \qquad Q_\lambda(u) := \int_{\mathbb{R}^N}\Big(|\nabla u|^2 - \lambda\frac{u^2}{|x|^2}\Big)dx.$$

For all $\lambda \in [0, \Lambda_N)$, Terracini proved in [2] that $S(\lambda)$ has positive radial minimizers:

$$Z_\mu^\lambda(x) := \frac{\mu^{-\frac{N-2}{2}}\big(2N(\Lambda_N - \lambda)/\sqrt{\Lambda_N}\big)^{\sqrt{\Lambda_N}/2}}{|\mu^{-1}x|^{\sqrt{\Lambda_N}-\sqrt{\Lambda_N-\lambda}}\big(1 + |\mu^{-1}x|^{2\sqrt{\Lambda_N-\lambda}/\sqrt{\Lambda_N}}\big)^{\sqrt{\Lambda_N}}}, \quad \forall\, \mu \in (0, +\infty). \tag{1.3}$$

If (H_2) and (H_3) hold, (1.2) implies that $Q_{\lambda_i(x)}(\cdot)^{1/2}$, $i = 1, 2$, are positive definite and the following best constants are well defined:

$$\bar{S}(\lambda_i) := \inf_{u\in D\backslash\{0\}}\frac{Q_{\lambda_i(x)}(u)}{\big(\int_{\mathbb{R}^N}|u|^{2^*}dx\big)^{\frac{2}{2^*}}}, \quad i = 1, 2, \tag{1.4}$$

$$\mathcal{A} = \mathcal{A}\big(\lambda_1, \lambda_2, \nu_1, \nu_2\big) := \inf_{(u,v)\in\mathcal{D}}\frac{Q_{\lambda_1(x)}(u) + Q_{\lambda_2(x)}(v)}{\big(\int_{\mathbb{R}^N}(\nu_1|u|^q + \nu_2|v|^q)^{\frac{2^*}{q}}dx\big)^{\frac{2}{2^*}}}, \tag{1.5}$$

where $\mathcal{D} := \mathbb{D}\setminus\{(0, 0)\}$. If ν_i, λ_i, $i = 1, 2$, are all constants, we rewrite $\mathcal{A}$ as $\mathcal{B}$:

$$\mathcal{B} = \mathcal{B}(\lambda_1, \lambda_2, \nu_1, \nu_2) := \inf_{(u,v)\in\mathcal{D}}\frac{Q_{\lambda_1}(u) + Q_{\lambda_2}(v)}{\big(\int_{\mathbb{R}^N}\big(\nu_1|u|^q + \nu_2|v|^q\big)^{\frac{2^*}{q}}dx\big)^{\frac{2}{2^*}}}. \tag{1.6}$$

Semilinear equations related to (1.2) have been studied by many authors recently (e.g. [2–6]). Elliptic systems related to (1.2) have also been studied very recently (e.g. [7–13]). In particular, the following Schrödinger system has been studied (e.g. [7,8,10,12]):

$$\begin{cases} -\Delta u - \lambda_1(x)\dfrac{u}{|x|^2} = |u|^{2^*-2}u + \dfrac{\nu\alpha}{2^*}h(x)|u|^{\alpha-2}|v|^\beta u, & \text{in } \mathbb{R}^N,\\ -\Delta v - \lambda_2(x)\dfrac{v}{|x|^2} = |v|^{2^*-2}v + \dfrac{\nu\beta}{2^*}h(x)|u|^{\alpha}|v|^{\beta-2} v, & \text{in } \mathbb{R}^N,\\ (u, v) \in D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N), \end{cases} \tag{1.7}$$

where $\alpha, \beta > 1$, $\alpha + \beta = 2^*$ and λ_1, λ_2, h satisfy the assumptions similar to (H_1)–(H_3). Some existence results of solutions to (1.7) were established under technical conditions.

Problem (1.1) is closely related to (1.7). For example, if $N = 4$, $q = \alpha = \beta = 2$, (1.1) is covered by (1.7). See also [11] and [13] for more results on (1.1). In this paper, we study the ground state solutions of (1.1), that is, those minimizing the Rayleigh quotient in (1.5). The main results of this paper are summarized in the following theorems and are new to the best of our knowledge.

Theorem 1.1. *Assume that* (H_1)–(H_3) *hold and* $1 < q < 2$.

(i) *Then there exists a constant* $\Lambda^* > 0$, *such that* (1.1) *has one positive ground state solution for all* $\lambda_1(x)$ *and* $\lambda_2(x)$ *satisfying* $\sup_{x\in\mathbb{R}^N}|\lambda_1(x) - \lambda_2(x)| < \Lambda^*$.

(ii) *Suppose that* $\nu_2 \le 1$. *Then there exists a constant* $\nu_1^* > 0$, *such that* (1.1) *has one positive ground state solution for all* $\nu_1 > \nu_1^*$.

(iii) *Suppose that* $\nu_1 \ge 1$. *Then there exists* $\nu_2^* > 0$, *such that* (1.1) *has one positive ground state solution for all* $\nu_2 \in (0, \nu_2^*)$.

Theorem 1.2. *Assume that* (H_1)–(H_3) *hold and* $2 \le q \le 2^*$.

(i) *Suppose that* $0 < \nu_1 < \nu_2$ *and either* $\bar{\lambda}_2 \le \lambda_1(0)$ *or* $\lambda_2(x) \equiv \lambda_2(0)$ *for all* $x \in \mathbb{R}^N$. *Then there exists a constant* $\Lambda^{**} > 0$, *such that* (1.1) *has one positive ground state solution for all* $\lambda_1(x), \lambda_2(x)$, *satisfying* $\sup_{x\in\mathbb{R}^N}|\lambda_1(x) - \lambda_2(x)| < \Lambda^{**}$.

(ii) *Suppose that* $\nu_1 \ge \nu_2 > 0$, $\lambda_1(x) \ge \lambda_2(x)$ *for all* $x \in \mathbb{R}^N$. *Then* $\mathcal{A}\big(\lambda_1, \lambda_2, \nu_1, \nu_2\big)$ *is only achieved by semi-trivial minimizers of the form* $(u, 0)$ *with* $u > 0$ *and* (1.1) *has one semi-trivial ground state of the form* $(\bar{u}, 0)$ *with* $\bar{u} > 0$.