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Some transport and diffusion processes on networks and their graph realizability

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ABSTRACT

In this paper we consider systems of transport and diffusion problems on one-dimensional domains coupled through transmission type boundary conditions at the endpoints and determine what types of such problems can be identified with respective problems on metric graphs. For the transport problem the answer is provided by a reformulation of a graph theoretic result characterizing line digraphs of a digraph, whereas in the case of diffusion the answer is provided by an algebraic characterization of matrices which are adjacency matrices of line graphs, which complements known results from graph theory and is particularly suitable for this application.

1. Introduction

Recently there has been an interest in dynamical problems on graphs, where some evolution operators, such as transport or diffusion, act on the edges of a graph and interact through nodes. One can mention here quantum graphs, see e.g. [1,2], diffusion on graphs in probabilistic context [3,4], transport problems, both linear and nonlinear [5–7], migrations [8], and several other applications discussed in e.g. [9,10]. In this note we shall focus on more general linear transport and diffusion problems posed on networks consisting of one dimensional domains, which are coupled through transmission conditions between an arbitrary selection of the endpoints of the domains. Well-posedness of such problems is studied in [11]; in this note we shall address the question under which conditions such problems can be realized by a flow or diffusion along the edges of a graph with transmission occurring at its nodes.

2. Transport problems

Models. We consider a multi-digraph (that is, a directed graph without loops but possibly with multiple edges) $\vec{G} = (V(\vec{G}), E(\vec{G})) = (\{v_1, \ldots, v_n\}, \{\vec{e}_1, \ldots, \vec{e}_m\})$ with n vertices $v_1, \ldots, v_n$ and m edges (arcs), $\vec{e}_1, \ldots, \vec{e}_m$. Denote $\mathcal{N} = \{1, \ldots, n\}$, $\mathcal{M} = \{1, \ldots, m\}$. We suppose that none of the vertices is isolated. Each edge is normalized so as to be identified with $[0, 1]$ with the head at 1 and the tail at 0. Then a simple transport on $\vec{G}$ with the Kirchhoff law of exchange of matter at the vertices

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can be written as [5,6,12],

$$\begin{cases} \partial_t u_i(x,t) + c_i \partial_x u_i(x,t) = 0, \quad x \in (0,1),\ t \geq 0,\ i \in \mathcal{M}, \\ u_i(x,0) = f_i(x), \\ \phi^-_{ki} c_i u_i(0,t) = w_{ki} \sum_{j=1}^{m} \phi^+_{kj} c_j u_j(1,t), \end{cases} \tag{2.1}$$

where $\Phi^- = (\phi^-_{ij})_{1\leq i\leq n, 1\leq j\leq m}$ and $\Phi^+ = (\phi^+_{ij})_{1\leq i\leq n, 1\leq j\leq m}$ are, respectively, the outgoing and incoming incidence matrices; that is, matrices with the entry ϕ^-_{ij} (resp. ϕ^+_{ij}) equal to 1 if there is edge $\vec{e}_j$ outgoing from (resp. incoming to) the vertex v_i, and zero otherwise. Further $\mathcal{C} = \mathrm{diag}\{c_j\}_{1\leq j\leq m}$ is the matrix of (positive) velocities of transport on each edge and $(w_{ij})_{1\leq i\leq n, 1\leq j\leq m}$ is the matrix of weights describing distribution of the incoming flow at the vertex v_i into the edges outgoing from it; it is a column stochastic matrix [12]. The non-negative vector $(f_i)_{i\in\mathcal{M}}$ describes the initial densities.

Under certain conditions (surjectivity of Φ^-) [5,12], (2.1) can be considered as the initial–boundary value problem

$$\mathbf{u}_t = A\mathbf{u}, \qquad \mathbf{u}(0) = \mathcal{K}\mathbf{u}(1), \qquad \mathbf{u}(0) = \mathbf{f}, \tag{2.2}$$

in $\mathbf{X} = (L_1([0,1]))^m$, where A is the realization of the expression $\mathsf{A} = (-c_j\partial_x)_{1\leq j\leq m}$ on the domain $D(A)$ consisting of functions $\mathbf{u} \in (W^1_1([0,1]))^m$ that satisfy the boundary condition. Here, $\mathcal{K} = (k_{ij})_{1\leq i,j\leq m}$ is given by $\mathcal{K} = \mathcal{C}^{-1}\mathcal{B}\mathcal{C}$. The column stochastic matrix $\mathcal{B}$ is the weighted transposed adjacency matrix for the line graph of $\vec{G}$, $\mathcal{B} = (\Phi^-_w)^T\Phi^+$, where Φ^-_w is obtained from Φ^- by replacing each nonzero ϕ^-_{ij} entry by w_{ij}. We also recall that the line graph $Q = L(\vec{G})$ of $\vec{G}$ is defined as $Q = (V(Q), E(Q)) = (E(\vec{G}), E(Q))$ and $E(Q)$ is the set of pairs (u, v) where $u, v \in E(\vec{G})$ and the head of u coincides with the tail of v.

However, the problems of the form (2.2) arise in a broader context, such as queuing theory [13], or population dynamics [14]. For instance, in the latter we consider a population of cells described by the density $u_j(x)$. Here x describes the degree of maturity of a cell ($0 \leq x \leq 1$); cells are born at $x = 0$ and divide at $x = 1$. Further, the cells are stratified into classes indexed by $j, j \in \mathcal{M}$, by the maturation rate which is assumed to only take discrete values c_j. Upon reaching maturity ($x = 1$), the cell divides producing offspring of arbitrary maturation velocities. The distribution of maturation velocities c_j of daughter cells from a mother with maturation velocity c_i is governed by a nonnegative matrix $\mathcal{B} = (b_{ij})_{1\leq j\leq m}$. Then $(u_j)_{j\in\mathcal{M}}$ satisfies

$$\begin{aligned} &\partial_t u_j(x,t) + v_j\partial_x u_j(x,t) = -\mu_j u_j(x,t), \quad x \in (0,1), t \geq 0, j \in \mathcal{M}, \\ &u_j(x,0) = \mathring{u}_j(x), \\ &v_j u_j(0,t) = \sum_{i\in\mathcal{M}} b_{ji} v_i u_i(1,t), \end{aligned} \tag{2.3}$$

where $\mu_j \geq 0$ is the death rate and the nonnegative vector $\mathring{u}$ describes the initial population. We observe that, contrary to the previous case, $\mathcal{B}$ need not be column stochastic since the mother cell can produce many offspring. In fact, $\mathcal{B}$ can be an arbitrary nonnegative matrix and thus, as we shall see later, despite formal similarity with (2.1), in general (2.3) does not describe any transport process on a graph.

Well-posedness of (2.2) in a general context ($\mathcal{K}$ even need not to be positive) is studied in [11] by methods developed in [5]. Here we address a natural question: under what conditions system (2.2) with general matrix $\mathcal{K}$ describes a graph transport model; that is, such that the exchange between subgroups only can occur if these are 'physically' connected by a node.

Graph-realizability of the models. It is clear that for (2.2) to represent (2.1), $\mathcal{K}$ must be nonnegative. For nonnegative $\mathcal{K}$ we introduce the matrix $\mathbb{K} = (\kappa_{ij})_{1\leq i,j\leq m}$ with $\kappa_{ij} = 1$ if $k_{ij} \neq 0$ and $\kappa_{ij} = 0$ otherwise. Since the multiplication by the diagonal matrices $\mathcal{C}$ and $\mathcal{C}^{-1}$ does not alter the places at which non-zero and zero entries appear in a matrix, we see that if $\mathcal{K} = \mathcal{C}^{-1}\mathcal{B}\mathcal{C}$, then $\mathbb{K}$ is the transposed adjacency matrix of the line graph Q of $\vec{G}$ [12]; that is, $\kappa_{ij} = 1$ if and only if there is a vertex v_k which is the head of e_j and the tail of e_i. It also follows that $\mathbb{K} = (\Phi^-)^T\Phi^+$. Moreover, since $\mathcal{B}$ is column stochastic, we have

$$\sum_{i=1}^{m} c_i k_{ij} = c_j. \tag{2.4}$$

Conversely, it is clear that if $\mathbb{K}$ is the transposed adjacency matrix of a graph Q which is the line graph of a graph $\vec{G}$ and (2.4) is satisfied, then we can build on $\vec{G}$ the transport problem (2.1) for which $\mathcal{K} = \mathcal{C}\mathcal{B}\mathcal{C}^{-1}$. Indeed, as mentioned before, $\mathcal{B}$ will have non-zero entries exactly where $\mathcal{K}$ has them and thus it will be the adjacency matrix of the line graph of a graph G, whereas (2.4) will ensure that $\mathcal{B}$ is column stochastic. In other words, the following definition makes sense.

Definition 2.1. We say that problem (2.2) is graph realizable if there exists a directed multigraph graph $\vec{G}$ such that $\mathcal{K} = \mathcal{C}\mathcal{B}\mathcal{C}^{-1}$, where $\mathcal{B}$ is the transposed weighted adjacency matrix of the line graph of $\vec{G}$.