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Lax pair, auto-Bäcklund transformation and conservation law for a generalized variable-coefficient KdV equation with external-force term

Zhang Yuping [a,*], Liu Jing [a], Wei Guangmei [b]

[a] State Key Laboratory of Software Development Environment, School of Computer Science and Engineering, Beihang University, Beijing 100191, China

[b] LMIB and School of Mathematics and System Sciences, Beihang University, Beijing 100191, China

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ABSTRACT

A generalized variable-coefficient KdV equation with perturbed and external-force terms is investigated in this Letter. Lax pair, Riccati-type auto-Bäcklund transformation and Wahlquist–Estabrook-type auto-Bäcklund transformation (WE–BT) are constructed. Based on the WE–BT, the nonlinear superposition formula is obtained and an infinite number of conservation laws are derived recursively, then the analytic solutions are provided including periodic, one-soliton-like and two-soliton-like solutions with inhomogeneous coefficients, external-force term and eigenvalue.

1. Introduction

In soliton theory, such properties as soliton solution, Bäcklund transformation and Lax pair have been studied for nonlinear evolution equations (NLEEs) [1–3]. The Korteweg–de Vries (KdV) equation, as a prototypical NLEE, can describe the nonlinear phenomena in many situations, such as ion-acoustic soliton in plasmas, stratified internal solitary waves (ISWs) in the ocean, collision-free hydromagnetic waves, lattice dynamics, and the jams in the congested traffic [1,4–6]. When the media are inhomogeneous or the boundaries are nonuniform, the variable-coefficient NLEEs are able to model various situations more realistically than their constant-coefficient counterparts [2,7,8]. In this Letter we will investigate a generalized variable-coefficient KdV(gvcKdV) equation with the perturbed/dissipative and external-force terms as follows [9],

$$u_t + f(t)uu_x + g(t)u_{xxx} + c(t)u_x + l(t)u = h(t), \tag{1}$$

where u is a wave function of the scaled space coordinate x and time coordinate t, $f(t)$, $g(t)$, $c(t)$, $l(t)$ and $h(t)$ are the analytic functions of t, representing the nonlinear, dispersive, line-damping, dissipative coefficients and external-force term, respectively.

Many physical and mechanical situations governed by Eq. (1) with special coefficients have been seen, e.g., ISWs in the ocean, pressure pulses in fluid-filled tubes of special value in arterial dynamics, trapped quasi-one-dimensional

* Corresponding author.
E-mail addresses: zhyuping@buaa.edu.cn (Y. Zhang), gmwei@buaa.edu.cn (G. Wei).

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Bose–Einstein condensates, ion-acoustic solitary waves in plasmas and the effect of a bump on wave propagation in a fluid-filled elastic tube [1,5,8–10].

Some special cases of Eq. (1) have been investigated. For $c(t) = l(t) = h(t) = 0$, the variable-coefficient case has been studied with Bäcklund transformation and similarity reductions [11]. For $h(t) = 0$, Lax pair, two auto-Bäcklund transformations and two-soliton-like solution based on the Ablowitz–Kaup–Newell–Segur (AKNS) system have been presented [1]. For $c(t) = 0$, the Painlevé integrable condition and auto-Bäcklund transformation based on Painlevé truncated method have been provided [8], and the N-soliton solution has been derived by the bilinear method [9].

The Letter is arranged as follows. In Section 2, based on the extended AKNS system, the Lax pair of Eq. (1) will be constructed. In Section 3, Riccati-type and Wahlquist–Estabrook-type auto-Bäcklund transformations and nonlinear superposition formula will be provided, and the analytic solutions will be obtained including periodic, one-soliton-like and two-soliton-like solutions with the inhomogeneous coefficients, external-force term and eigenvalue. An infinite number of recursive conservation laws will be derived in Section 4. Finally, conclusion and discussion will be given in Section 5.

2. Lax pair

The Lax pair may assure the complete integrability of a NLEE, and can be used to obtain the multi-solitonic solutions by means of the Darboux transformation method and inverse scattering transform method [1,2,12,13]. To construct the Lax pair of Eq. (1), we introduce the functions $E(x, t)$, $W(x, t)$ and $G(x, t)$ in the AKNS system [1,2,13], and the Lax pair of Eq. (1) can be expressed as,

$$\Phi_x = U\Phi = \begin{pmatrix} \lambda & E(u+W) \\ G & -\lambda \end{pmatrix} \Phi, \tag{2}$$

$$\Phi_t = V\Phi = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \Phi, \tag{3}$$

where $\Phi = (\phi_1, \phi_2)^T$, λ is a parameter independent of x and t, $A(x, t, \lambda)$, $B(x, t, \lambda)$, and $C(x, t, \lambda)$ are functions of x and t, and U and V are two 2×2 null-trace matrices, which must satisfy the compatibility condition $U_t - V_x + UV - VU = 0$.

In the following procedure, we will stay with the constraint $f(t) = \alpha g(t)e^{\int l(t)dt}$, which is a special case of the Painlevé integrable condition of Eq. (1), with α an arbitrary constant [8].

Expanding $A(x, t, \lambda)$, $B(x, t, \lambda)$, and $C(x, t, \lambda)$ with respect to λ as

$$\begin{cases} A = a_0(x,t) + a_1(x,t)\lambda + a_2(x,t)\lambda^2 + a_3(x,t)\lambda^3, \\ B = b_0(x,t) + b_1(x,t)\lambda + b_2(x,t)\lambda^2, \\ C = c_0(x,t) + c_1(x,t)\lambda + c_2(x,t)\lambda^2, \end{cases}$$

and substituting them into the compatibility condition yield

$$E(x,t) = e^{kx+\frac{1}{2}\int l(t)dt}, \tag{4}$$

$$G(x,t) = -\frac{\alpha}{6}e^{-kx+\frac{1}{2}\int l(t)dt}, \tag{5}$$

$$W(t) = -e^{\int l(t)dt}\left(C_1 + \int h(t)e^{\int l(t)dt}dt\right), \tag{6}$$

$$\begin{aligned} A \ = \ & -4g(t)\lambda^3 + 6\,kg(t)\lambda^2 - \left[\frac{\alpha g(t)}{3}e^{\int l(t)dt}(u+W) - \alpha g(t)e^{\int l(t)dt}W + 3k^2 g(t) + l(t)\right]\lambda \\ & + \frac{\alpha k}{6}g(t)e^{\int l(t)dt}(u+W) - \frac{\alpha}{6}g(t)e^{\int l(t)dt}u_x - \frac{l(t)}{4} - \frac{k\alpha}{2}g(t)e^{\int l(t)dt}W + \frac{k^3}{2}g(t), \end{aligned} \tag{7}$$

$$\begin{aligned} B \ = \ & -4g(t)e^{-kx+\frac{1}{2}\int l(t)dt}(u+W)\lambda^2 + [4\,kg(t)(u+W) - 2gu_x]e^{kx+\frac{1}{2}\int l(t)dt}\lambda \\ & - \frac{\alpha g(t)}{3}e^{kx+\frac{3}{2}\int l(t)dt}(u+W)^2 + [\alpha g(t)e^{\int l(t)dt}W - k^2 g(t) - c(t)]e^{kx+\frac{1}{2}\int l(t)dt}(u+W) \\ & + (ku_x - u_{xx})g(t)e^{kx+\frac{1}{2}\int l(t)dt}, \end{aligned} \tag{8}$$

$$\begin{aligned} C \ = \ & \frac{2\alpha}{3}g(t)e^{-kx+\frac{1}{2}\int l(t)dt}\lambda^2 - \frac{2\alpha k}{3}g(t)e^{-kx+\frac{1}{2}\int l(t)dt}\lambda + \frac{\alpha^2}{18}g(t)e^{-kx+\frac{3}{2}\int l(t)dt}(u+W) \\ & + \frac{\alpha}{6}[-\alpha g(t)e^{\int l(t)dt}W + k^2 g(t) + c(t)]e^{-kx+\frac{1}{2}\int l(t)dt}, \end{aligned} \tag{9}$$

where k and C_1 are two arbitrary parameters. It can be verified that the compatibility condition $\Phi_{xt} = \Phi_{tx}$ with Eqs. (4)–(9) leads to Eq. (1). It is noted that Eqs. (2) and (3) with $k = 0$, $C_1 = 0$, $h(t) = 0$ are the Lax pair for Eq. (1) with $h(t) = 0$, which agree with the corresponding results in Ref. [1].