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The variational iteration method is a special case of the homotopy analysis method

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ABSTRACT

In the present paper, we demonstrate that the variational iteration method (and all of its optimal analogues) are specific cases of the more general homotopy analysis method. To do so, we derive the variational iteration method starting with the homotopy analysis method. The optimal variational iteration method, which also appears in the literature, can be described completely within the context of the optimal homotopy analysis method. Alternately, the optimal homotopy analysis method can be used to construct more general iterative methods of the same ilk, which is potentially useful for solving partial differential equations.

1. Introduction

In the past, it has been shown that various analytical methods which appear in the literature, such as the homotopy perturbation method (HPM) [1] and the Adomian decomposition method (ADM) [2], are essentially special cases of the homotopy analysis method (HAM) [3], as the latter is more robust and general. It has also been shown that there are clear cases where the former approaches fail, while the homotopy analysis method approach succeeds at obtaining a reasonable analytical approximation to certain nonlinear initial or boundary value problems. Indeed, since the HAM allows one to control and adjust the convergence properties of the solution, it can succeed in solving some problems where these other methods fail. For instance, it has been shown by Allan [4] that the ADM can be directly derived from the HAM. It was later shown that, since the ADM does not have any adjustable convergence control parameters, convergence properties of its solutions are often inferior to those of the solutions obtained via the HAM [5], so, the HAM solution can remain valid over a larger region. Similarly, the HPM has been shown by Liao [6] to be a special case of the HAM when the convergence control parameter in the HAM is set equal to minus one. Due to this restriction, it has been shown [7,8] that the HPM can yield divergent solutions for problems where the HAM yields accurate and convergent solutions.

One recently popular approach for solving nonlinear differential equations is the variational iteration method (VIM) [9]. However, since the VIM appears, at least superficially, to involve an iterative scheme based around an integral, there has not been a theoretical study discussing the approach in the context of the HAM. In the present paper, we demonstrate that the VIM, in all of its various forms, can be derived directly from the HAM. In doing so, we gain two things. The first is the knowledge that the theory and techniques for solving nonlinear differential equations through the HAM can deal with any problem that might be solved using the VIM. This suggests that the application of VIM is not needed, since it does not offer anything new that we cannot obtain from HAM. The second benefit is that, by studying the method in the wider context of the HAM, we are actually able to obtain more general iteration methods which may exhibit better convergence properties

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than those of the standard or optimal VIMs found in the literature, as the standard VIM has been shown to yield divergent solutions in some cases where the HAM gives accurate and convergent solutions [7]. As a consequence of the fact that we can construct more general iterative methods by use of the HAM, one may derive a non-trivial iteration method for partial differential equations by considering auxiliary linear operators that depend on multiple variables.

In Section 2, we derive the VIM from the HAM. We then obtain more general iteration methods, by use of more carefully selected auxiliary linear operators in the HAM. We mention optimal iteration techniques which minimize error, demonstrating that any so-called optimal VIMs, such as those discussed in [10,11], are really special cases of the optimal HAM. We discuss how both the convergence control parameter and the auxiliary linear operator in the HAM can be selected in order to obtain iterative methods with minimal residual errors. Some concluding remarks are then given in Section 3.

2. Derivation of VIMs from the HAM

We shall begin by considering the HAM in the standard way [3,12–14], and from this method we shall derive the most general form of the VIM. Consider the ordinary differential equation

$$L[u] + N[u] = f(x), \tag{1}$$

where $u = u(x)$ is some unknown function to be determined, L is a linear differential operator, N is a nonlinear differential operator, and $f(x)$ is a given source term, or, inhomogeneity. Keeping with the general flavor of the VIM, we partition the linear and nonlinear parts of the differential equation, although one could group them as a single operator with no loss of generality.

Let us consider the linear homotopy between the nonlinear problem (1) and an auxiliary linear problem $\mathcal{L}[u] = 0$ given by

$$\mathcal{H}_q \equiv (1-q)\mathcal{L}[u] - qhH(x)\left(L[u] + N[u] - f(x)\right). \tag{2}$$

Here, $q \in [0, 1]$ is the embedding parameter, $\mathcal{L}$ is the auxiliary differential linear operator, $H(x)$ is the so-called auxiliary function, and $h \neq 0$ is the convergence control parameter. Typically, we set $\mathcal{H}_q$ identically equal to zero. Doing so, when $q = 0$ we obtain a linear problem $\mathcal{L}[u] = 0$, while when $q = 1$ we obtain the original nonlinear problem, $L[u] + N[u] - f(x) = 0$.

One makes the assumption of a solution which depends as a series on the embedding parameter, q, so that

$$u(x) = u_0(x) + \sum_{n=1}^{\infty} u_n(x) q^n. \tag{3}$$

At $q = 0$, we should have the solution to the auxiliary linear problem, whereas when $q = 1$, we hopefully obtain a solution to the nonlinear problem, provided that this infinite series of functions converges. Here we shall assume that the behavior of this series is sufficiently nice as $q \to 1^-$.

Realistically, one truncates this series expansion when actually working with nonlinear equations, obtaining a $k+1$-term partial sum expansion

$$\hat{u}_k(x) = u_0(x) + \sum_{n=1}^{k} u_n(x). \tag{4}$$

This type of expansion is reasonable when the terms that were discarded are sufficiently small. As we discuss later in this section, it is advantageous to determine exactly if this approximation is accurate. As we shall see, this accuracy will often depend on the value of h, the convergence control parameter.

If we place (3) into the homotopy, which is then set identically equal to zero, we obtain successively a system of equations by which we may recover the unknown functions in (3), to wit,

$$\mathcal{L}[u_0] = 0, \tag{5}$$

$$\mathcal{L}[u_{n+1} - u_n] = hH(x)\left(L[u_n] + N_n[\hat{u}_n] - \delta_{0,n} f(x)\right) \quad \text{for} \quad n = 0, 1, 2, \ldots. \tag{6}$$

Here, N_n is the nth term in the expansion of $N[u]$ in q, where u is taken as the series (3). The terms with $\delta_{0,n}$ are used to remove the leading term (since $\mathcal{L}[u_0]$) and add the inhomogeneous contribution which only appears at lowest order when $n = 0$ (i.e., when we are solving for $n = 1$). Note that if we were to pick u_0 such that (5) was not satisfied, then (7) would contain a contribution from $u_0(x)$ outside the integral, when $n = 1$.

These terms can be computed recursively for a specific form of N, and the term N_n will always involve terms u_0 through u_n, that is, all of the terms in the partial sum $\hat{u}_n$, but never higher-order terms. Assuming we can solve these differential equations (since one is free to select their own choice of $\mathcal{L}$, one typically chooses $\mathcal{L}$ so that the system can indeed be solved), we successively recover the functions $u_0(x)$, $u_1(x)$, and so on. Note that $u_n(x)$ will depend in some way on the parameter h, for $n \geq 1$; for now we suppress this dependence, although we shall say more on this in the next section. We typically solve u_0 subject to any boundary or initial data, and then solve for the higher-order contributions with homogeneous boundary or initial data since such data does not typically depend on the embedding parameter, q.