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Commuting solutions of the Yang–Baxter matrix equation

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ABSTRACT

Let A be a square matrix with some special Jordan forms. When A is nonsingular, we find all the solutions of the quadratic matrix equation $AXA = XAX$, which commute with A. We also find infinitely many solutions commuting with A, depending on several parameters, when A is singular.

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1. Introduction

We are interested in finding the solutions of the quadratic matrix equation

$$AXA = XAX, \tag{1}$$

where both the given A and the unknown X are $n \times n$ complex matrices. This equation is called here the *Yang–Baxter matrix equation* since it is similar in format to the one in the parameter-free Yang–Baxter equation. In the fields of statistical mechanics, knot theory, braid groups, and quantum theory, the Yang–Baxter equation has been a hot research topic [1], but in matrix theory, this special quadratic matrix equation has not been systematically studied yet. One reason is the fact that finding all the solutions is a difficult task.

Some results on solving the Yang–Baxter matrix equation have been obtained in [2,3] with various techniques, such as Brouwer's fixed point theorem, the mean ergodic theorem, and the spectral theorem.

In a recent paper [4], it was proved that a matrix B is a solution of (1) if it satisfies the condition $AB = BA = B^2$. A solution of (1) that commutes with A is called a *commuting solution*. Based on this general result, several explicit commuting solutions were obtained when A has only one eigenvalue with some particular Jordan blocks.

If A is a block diagonal matrix such that its diagonal blocks are of the same structure as studied in [4], then all the block diagonal matrices of the same size, whose diagonal blocks are the solutions of (1) from [4] associated with the diagonal blocks of A, must be commuting solutions of (1). The question is whether we can find all commuting solutions. In this note we answer this question when A is nonsingular with some special Jordan structure. Using a general result in matrix theory, we could show that when there are several eigenvalues, it is possible to find commuting solutions since that system turns out to be a block diagonal one.

In the next section we prove two theorems on communing solutions of (1) after establishing a preliminary result.

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2. Commuting solutions

In order to find all commuting solutions of the Yang–Baxter matrix equation, we need several lemmas on solving the linear matrix equation

$$HX = XK, \tag{2}$$

where H is an $h \times h$ matrix and the matrix K is $k \times k$.

We denote a $k \times k$ Jordan block with eigenvalue λ by $J_k(\lambda)$, that is, $J_k(\lambda)$ is the zero matrix with the diagonal and super-diagonal replaced with all λ's and all 1's respectively. We also write $J_k(\lambda)$ as $J(\lambda)$ if its size is not specified.

The following result was proved, for example, in [5]. But here we present a simpler proof.

Lemma 2.1. *Let X be a $h \times k$ matrix and let c be a nonzero number. Then $X = 0$ is the only solution of the following equation*

$$XJ_k(0) - J_h(0)X = cX. \tag{3}$$

Proof. It is easy to show that $\big(J_k(0)\big)^k = 0$. Let X be any solution of (3). Using the fact $XJ_k(0) = \big(cI + J_h(0)\big)X$ repeatedly for k times leads to

$$\begin{aligned} 0 &= X\big(J_k(0)\big)^k = XJ_k(0) \cdot \big(J_k(0)\big)^{k-1} = \big(cI + J_h(0)\big)X \cdot \big(J_k(0)\big)^{k-1} \\ &= \cdots = \big(cI + J_h(0)\big)^{k-1}X \cdot J_k(0) = \big(cI + J_h(0)\big)^k X. \end{aligned}$$

Clearly, $cI + J_h(0)$ is a nonsingular matrix, so is $\big(cI + J_h(0)\big)^k$. Multiplying $\big(cI + J_h(0)\big)^{-k}$ from left to the above equality, we have $X = 0$. □

Lemma 2.2. *Suppose that H and K have no common eigenvalues. Then the matrix equation (2) has only the trivial solution $X = 0$.*

Proof. Let J_H and J_K be the Jordan forms of H and K, respectively. Then there are nonsingular matrices P and Q such that $H = PJ_HP^{-1}$ and $K = QJ_KQ^{-1}$. The equation $HX = XK$ for X is equivalent to the equation $J_HY = YJ_K$ for Y with $Y = P^{-1}XQ$. Write

$$J_H = \mathrm{diag}(J_{h_1}(\lambda_1), \ldots, J_{h_r}(\lambda_r)) \quad \text{and} \quad J_K = \mathrm{diag}(J_{k_1}(\mu_1), \ldots, J_{k_s}(\mu_s)).$$

Partitioning $Y = [Y_{ij}]$ accordingly as an $r \times s$ block matrix, and multiplying out the equation $J_HY = YJ_K$ via block matrix multiplications, we obtain a family of rs matrix equations

$$J_{h_i}(\lambda_i)Y_{ij} = Y_{ij}J_{k_j}(\mu_j), \quad i = 1, \ldots, r;\ j = 1, \ldots, s.$$

Since $J_{h_i}(\lambda_i) = \lambda_i I + J_{h_i}(0)$ and $J_{k_j}(\mu_j) = \mu_j I + J_{k_j}(0)$, where I is an identity matrix of suitable size, the above equations can be written as

$$(\lambda_i - \mu_j)Y_{ij} = Y_{ij}J_{k_j}(0) - J_{h_i}(0)Y_{ij}, \quad \forall\, i, j.$$

Now the assumption that $\lambda_i \neq \mu_j$ for $i = 1, \ldots, r$ and $j = 1, \ldots, s$ and Lemma 2.1 imply that the above equations have only zero solutions. Hence $Y = 0$, so $X = PYQ^{-1} = 0$. □

Remark 2.1. When H and K have common eigenvalues, the structure of the solutions to (2) was obtained in [5] (Theorem 5.16).

Lemma 2.3. *Let a block diagonal matrix $H = \mathrm{diag}(H_1, \ldots, H_t)$ be such that no pairs H_i and H_j have common eigenvalues for $i \neq j$. If $HK = KH$, then $K = \mathrm{diag}(K_1, \ldots, K_t)$, where K_i has the same size as H_i for $i = 1, \ldots, t$.*

Proof. Partitioning $K = [K_{ij}]$ according to the block structure of H, we multiply out the equation

$$\mathrm{diag}(H_1, \ldots, H_t)[K_{ij}] = [K_{ij}]\mathrm{diag}(H_1, \ldots, H_t)$$

to give the equations

$$H_iK_{ij} = K_{ij}H_j, \quad i, j = 1, \ldots, t.$$

Now, for $i \neq j$, since H_i and H_j have no common eigenvalues, Lemma 2.2 guarantees that $K_{ij} = 0$. This shows that K is block diagonal.

Now we apply Lemma 2.3 to solving the Yang–Baxter matrix equation (1) for some cases. It is difficult to find all the solutions of the nonlinear matrix equation for a general matrix A since the solutions form disconnected manifolds of complicated structure, which can be seen even for the 2×2 case from [4]. Here our focus is to find the commuting solutions of (1), that is, the matrices X that satisfy the two equations $AX = XA$ and $AXA = XAX$.

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