Applied Mathematics Letters 44 (2015) 21–25

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Pulses and waves for a bistable nonlocal reaction–diffusion equation

V. Volpert

Institut Camille Jordan, University Lyon 1, UMR 5208 CNRS 69622 Villeurbanne, France

ARTICLE INFO

Article history:
Received 18 August 2014
Received in revised form 14 December 2014
Accepted 14 December 2014
Available online 8 January 2015

Keywords:
Nonlocal reaction–diffusion equations
Waves
Pulses

ABSTRACT

A bistable nonlocal reaction–diffusion equation is studied. Solutions in the form of simple and periodic travelling waves, single and multiple pulses are observed in numerical simulations. Successive transitions from simple waves to periodic waves and to stable pulses are described.

1. Introduction

Nonlocal reaction–diffusion equations

$$\frac{\partial u}{\partial t} = d\,\frac{\partial^2 u}{\partial x^2} + au^2(1 - J(u)) - \sigma u, \tag{1.1}$$

where

$$J(u) = \int_{-\infty}^{\infty} \Phi(x-y)u(y,t)dy,$$

describes various biological phenomena such as emergence and evolution of biological species and the process of speciation in a more general context [1,2]. An important property of such equations is that they have solutions in the form of periodic travelling waves [3–5]. Such solutions do not exist for the usual (scalar) reaction–diffusion equations. In this work we present a new type of solutions of this equation, single and multiple pulses, and show how they are related to periodic travelling waves. We will consider the kernel $\Phi(x)$ in two different forms. In the first case, we set $\Phi = \phi$, where

$$\phi(x) = \begin{cases} 1/(2N), & |x| < N \\ 0, & |x| \geq N \end{cases}$$

is a step-wise constant function. In the limit of small N we obtain the reaction–diffusion equation

$$\frac{\partial u}{\partial t} = d\,\frac{\partial^2 u}{\partial x^2} + au^2(1 - u) - \sigma u. \tag{1.2}$$

It is well known that it can have travelling wave solutions and solutions in the form of stationary pulses, that is positive solutions decaying at infinity. Travelling waves are asymptotically stable with shift while pulses are unstable. Existence of waves

E-mail address: volpert@math.univ-lyon1.fr.

http://dx.doi.org/10.1016/j.aml.2014.12.011
0893-9659/

for the nonlocal equation (1.1) is proved for sufficiently small N [3,4]. Existence of waves for some other nonlocal equations can be proved without the assumption that the support of the kernel is small [6,7]. Existence of pulses can also be proved for the kernels with a small support. Moreover, travelling waves are stable if N is small enough and pulses are unstable.

The second limiting case of Eq. (1.1) is that of large N. Instead of the kernel $\phi(x)$ we now consider the kernel $\Phi(x) = \psi(x)$, where $\psi(x) = 1$ for $|x| < N$ and 0 for $|x| \geq N$. The limiting equation becomes

$$\frac{\partial u}{\partial t} = d\,\frac{\partial^2 u}{\partial x^2} + au^2(1 - I(u)) - \sigma u, \quad I(u) = \int_{-\infty}^{\infty} u(y,t)dy. \tag{1.3}$$

This equation does not have travelling waves but it has stationary pulses. The integral term in this equation can make them stable [1]. Therefore we can expect that stable pulses also exist for Eq. (1.1) if N is sufficiently large [8].

Thus, for sufficiently small N nonlocal equation (1.1) has stable waves and unstable pulses. For sufficiently large N, it can have stable pulses but there are no waves. In this work we will study transition of solutions of this equation from stable waves to stable pulses as N increases.

2. Travelling waves

If $\sigma/a < 1/4$, then Eq. (1.1) has three homogeneous in space stationary solutions, $u_+ = 0$, and two other solutions u_0 and u_-, $u_0 < u_-$, of the equation $u(1-u) = \sigma/a$. The homogeneous in space stationary solution $u = u_-$ of Eq. (1.1) can lose its stability resulting in appearance of periodic in space solutions [9–11]. If we consider a localized in space perturbation of this homogeneous in space solution, then it propagates as a periodic wave. The speed and the amplitude of this wave depend on parameters. In the linear approximation, the speed can be estimated through the maximal positive eigenvalue [12].

Let c_0 be the speed of the wave with the limits $w(\pm\infty) = u_\pm$, which exists at least for sufficiently small N, and c_p be the average speed of the periodic wave which effectuates transition from u_- to the periodic in space stationary solution. If the support N of the kernel $\phi(x)$ decreases and tends to the critical value for which this periodic stationary solution bifurcates from the homogeneous in space solution, then the speed of the periodic wave converges to zero. Therefore for sufficiently small N, $c_0 > c_p$, and the $[u_+, u_-]$-wave runs away from the periodic wave.

Fig. 1 shows different regimes of wave propagation. If the solution u_- is stable, then there is a $[u_+, u_-]$-wave with the limits $u_\pm$ at $\pm\infty$. It is not monotone with respect to x. The green lines in the left figure show the position of the maxima of solution. They move altogether with the wave front. If N is greater than the critical value $N_c \approx 3.6$, then the homogeneous in space stationary solution becomes unstable and a periodic in space stationary solution emerges behind the $[u_+, u_-]$-wave. If N is close to the critical value, then the amplitude and the speed of propagation of the periodic wave are small. It propagates slower than the $[u_+, u_-]$-wave (Fig. 1, middle). For a greater N, they propagate with the same speed but the periodic wave stays at some distance behind the $[u_+, u_-]$-wave (Fig. 1, right). Its influence is exponentially small, and the $[u_+, u_-]$-wave can still be considered as having a constant speed and profile. Finally for sufficiently large values of N, the two waves merge forming a single periodic wave (Fig. 2, middle).

3. Single and multiple pulses

3.1. Unstable pulses

Consider the equation

$$dw'' + aw^2(1 - w) - \sigma w = 0. \tag{3.1}$$

Set $F(w) = aw^2(1-w) - \sigma w$. If $\int_0^{u_-} F(u)du > 0$, then it has a positive solution $w_0(x)$ which decays at infinity. This solution is unstable since the corresponding linearized operator has a positive eigenvalue [1]. Along with Eq. (3.1) we consider the corresponding nonlocal equation

$$dw'' + aw^2(1 - J(w)) - \sigma w = 0, \quad J(w) = \int_{-\infty}^{\infty} \phi(x-y)w(y)dy. \tag{3.2}$$

It can be proved by the perturbation technique similar to travelling waves [3,4] that for all N sufficiently small there exists a pulse solution of this equation. The proof uses the implicit function theorem and the spectral properties of the linearized operator. This pulse solution is unstable for sufficiently small N.

3.2. Stable pulses

Next, consider the equation

$$\frac{\partial u}{\partial t} = d\,\frac{\partial^2 u}{\partial x^2} + au^2(1 - K(u)) - \sigma u, \quad K(u) = \int_{-\infty}^{\infty} \psi(x-y)u(y,t)dy, \tag{3.3}$$