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A new recursive algorithm for inverting general k-tridiagonal matrices

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ABSTRACT

In the present article we give a new breakdown-free recursive algorithm for inverting general k-tridiagonal matrices without imposing any simplifying assumptions. The implementation of the algorithm in Computer Algebra Systems (CAS) such as Maple, Mathematica and Macsyma is straightforward. Two illustrative examples are given.

1. Introduction

The general tridiagonal matrix $T = (t_{ij})_{i,j=1}^{n}$ in which $t_{ij} = 0$ for $|i - j| > 1$ can be written in the form:

$$T = \begin{bmatrix} d_1 & a_1 & 0 & \dots & 0 \\ b_1 & d_2 & a_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & b_{n-2} & d_{n-1} & a_{n-1} \\ 0 & \dots & 0 & b_{n-1} & d_n \end{bmatrix}. \tag{1}$$

Tridiagonal matrices frequently appear in a variety of applications such as parallel computing, cubic spline interpolation, telecommunication system analysis, and in numerous other fields of science and engineering. In many of these areas inversion of tridiagonal matrices is required. The interested reader may refer to [1–12] and the references therein.

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A more general tridiagonal matrix is the k-tridiagonal matrix $T_n^{(k)} = (\hat{t}_{ij})_{i,j=1}^n$ which can be written in the form:

$$T_n^{(k)} = \begin{bmatrix} d_1 & 0 & \dots & 0 & a_1 & 0 & \dots & 0 \\ 0 & d_2 & 0 & \dots & 0 & a_2 & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \dots & \ddots & \ddots & 0 \\ 0 & \dots & \ddots & \ddots & \ddots & \dots & \ddots & a_{n-k} \\ b_1 & 0 & \dots & \ddots & \ddots & \ddots & \dots & 0 \\ 0 & b_2 & \ddots & \dots & \ddots & \ddots & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \dots & 0 & d_{n-1} & 0 \\ 0 & \dots & 0 & b_{n-k} & 0 & \dots & 0 & d_n \end{bmatrix}. \tag{2}$$

For the matrix $T_n^{(k)}$ in (2), $\hat{t}_{ij} = 0$ for all $i, j = 1, 2, \dots, n$ except for $|i - j| = 0$ or k, where $k \in \{1, 2, \dots, n-1\}$. For $k \geq n$, the matrix $T_n^{(k)}$ is a diagonal matrix and the case $k = 1$ gives the ordinary tridiagonal matrix in (1). In [13], it has been found that the k-tridiagonal matrix plays an important role in describing generalized k-Fibonacci numbers. Moreover, the authors in [14,15] computed integer powers of some special types of these matrices by exploiting some properties of Chebyshev polynomials. In [16–19] the authors investigated k-tridiagonal matrices in some details. There is no need to store the zero elements of $T_n^{(k)}$. The nonzero elements of the matrix in (2) can be stored in $3n-2k$ memory locations by using three vectors $\mathbf{a} = [a_1, a_2, \dots, a_{n-k}]$, $\mathbf{b} = [b_1, b_2, \dots, b_{n-k}]$ and $\mathbf{d} = [d_1, d_2, \dots, d_n]$. This is always a good habit in computation in order to save memory space.

Throughout this paper, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. Also, the word 'simplify' means simplify the expression under consideration to its simplest rational form.

The organization of the paper is as follows. In Section 2, a new breakdown-free symbolic computational algorithm is presented. Two illustrative examples are given in Section 3.

2. Main results

In this section we are going to consider the construction of a new computational algorithm for inverting any nonsingular k-tridiagonal matrix. For this purpose it is helpful to introduce an n-component vector $\mathbf{c} = [c_1, c_2, \dots, c_n]$ whose components are given by [20]:

$$c_i = \begin{cases} d_i, & \text{for } i = 1, 2, \dots, k \\ d_i - \dfrac{b_{i-k}\, a_{i-k}}{c_{i-k}}, & \text{for } i = k+1, k+2, \dots, n. \end{cases} \tag{3}$$

With the help of the vector $\mathbf{c}$ in (3), we may formulate the following result:

Lemma 2.1 ([21,22]). *Let $T_n^{(k)}$ be a k-tridiagonal matrix in (2) for which $c_i \neq 0$, for $i = 1, 2, \dots, n$. Then the Doolittle LU factorization of $T_n^{(k)}$ is given by:*

$$T_n^{(k)} = L_n^{(k)}\, U_n^{(k)}, \tag{4}$$

where

$$L_n^{(k)} = \begin{bmatrix} 1 & 0 & \dots & & & & \dots & 0 \\ 0 & 1 & \ddots & & & & & \vdots \\ \vdots & 0 & \ddots & \ddots & & & & \\ 0 & \vdots & \ddots & \ddots & \ddots & & & \\ \frac{b_1}{c_1} & \ddots & \vdots & \ddots & \ddots & \ddots & & \\ 0 & \frac{b_2}{c_2} & \ddots & \dots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \dots & 0 & \ddots & 0 \\ 0 & \dots & 0 & \frac{b_{n-k}}{c_{n-k}} & 0 & \dots & 0 & 1 \end{bmatrix}, \quad U_n^{(k)} = \begin{bmatrix} c_1 & 0 & \dots & 0 & a_1 & 0 & \dots & 0 \\ 0 & c_2 & 0 & \dots & 0 & a_2 & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \dots & \ddots & \ddots & 0 \\ & & \ddots & \ddots & \ddots & \dots & \ddots & a_{n-k} \\ & & & \ddots & \ddots & \ddots & \dots & 0 \\ & & & & \ddots & \ddots & 0 & \vdots \\ \vdots & & & & & \ddots & c_{n-1} & 0 \\ 0 & \dots & & & & \dots & 0 & c_n \end{bmatrix}, \tag{5}$$