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Pricing Parisian down-and-in options

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ABSTRACT

In this paper, we price American-style Parisian down-and-in call options under the Black–Scholes framework. Usually, pricing an American-style option is much more difficult than pricing its European-style counterpart because of the appearance of the optimal exercise boundary in the former. Fortunately, the optimal exercise boundary associated with an American-style Parisian knock-in option only appears *implicitly* in its pricing partial differential equation (PDE) systems, instead of *explicitly* as in the case of an American-style Parisian knock-out option. We also recognize that the "moving window" technique developed by Zhu and Chen (2013) for pricing European-style Parisian up-and-out call options can be adopted to price American-style Parisian knock-in options as well. In particular, we obtain a simple analytical solution for American-style Parisian down-and-in call options and our new formula is written in terms of four double integrals, which can be easily computed numerically.

1. Introduction

Barrier options are common path-dependent options traded in financial markets. One of reasons for the popularity of barrier options is that they provide a more flexible and cheaper way for hedging and speculating than vanilla options because the holders of barrier options only pay a premium for scenarios they perceive as likely. The "one touch" breaching barrier, however, may have an undesirable feature of suddenly losing all proceeds (knock-out options) or suddenly receiving the embedded options (knock-in options) if the price of the underlying asset momentarily touches the asset barrier, no matter how briefly the breaching occurs. This opens up the possibility of market practitioners deliberately manipulating the underlying asset price to force the cancellation or activation of the option. To partially remedy such a drawback, Parisian options are introduced, with a unique feature that the underlying asset price has to continually stay above or below the asset barrier for a prescribed amount of time before the knock-out or knock-in feature is activated. This extended trigger clause can also be found in some derivative contracts, such as callable convertible bonds and executive warrants [1]. It is also worthwhile to note that Parisian options can be a useful tool in corporate finance [2].

Like the relationship between a vanilla American option and its European counterpart, the valuation problem of an American-style Parisian option, in general, is much more difficult than that of its European-style counterpart. While a closed-form solution of the latter has already been found by Zhu and Chen [3], a closed-form solution of the former only exists for the perpetual knock-in case [4]. The extra difficulty of pricing an American-style Parisian knock-out option, in comparison with its European-style counterpart, mainly stems from the complexity of the determination of the optimal exercise boundary, which is a three-dimensional (3-D) surface, instead of a two-dimensional (2-D) curve as for the case of a vanilla American option. However, this difficulty disappears in the valuation of an American-style Parisian knock-in option. This is because the option holder cannot do or decide anything until the knock-in option is activated and once the "knock-in" feature is

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activated, the optimal exercise boundary of the option is the same as that of the embedded vanilla American option, the calculation of which has been thoroughly studied in the literature [5–16]. This suggests that the valuation of American-style Parisian knock-in options should be similar to that of their European-style counterparts and thus a simple analytical solution can be achieved. This paper aims to derive an explicit analytical solution for American-style Parisian down-and-in call options. With a growing demand for trading exotic options in today's finance industry, our solution procedures may lead to the development of pricing formulae for other exotic derivatives, such as the Edokko options introduced by Fujita and Miura [17], which are generalizations of both Parisian and delayed barrier options.

The paper is organized as follows. In Section 2, we introduce the PDE systems governing the price of an American-style Parisian down-and-in call option. In Section 3, the solution procedure is presented. Conclusion is given in the last section.

2. The PDE systems

By definition, an American-style Parisian down-and-in call option will be knocked in and become the embedded vanilla American call option if the underlying asset price continually stays below the barrier $\bar{S}$ for a prescribed time period $\bar{J}$. Otherwise, the Parisian down-and-in call option will expire worthless.

For some extreme values of $\bar{S}$ and $\bar{J}$, one can easily observe that an American-style Parisian down-and-in call option becomes worthless or degenerates to either a one-touch barrier option or a vanilla option. For other non-degenerate cases, the price of an American-style Parisian down-and-in call option is, however, not trivial. It depends on the underlying asset price S, the current time t and the barrier time J, in addition to other parameters such as the volatility rate σ, the risk-free interest rate r and the expiry time T. We now assume that the underlying asset price S with a continuous dividend yield D follows a lognormal Brownian motion given by

$$dS = (r - D)Sdt + \sigma SdZ, \tag{2.1}$$

where Z is a standard Brownian motion.

Based on the same financial arguments in [3], the pricing domains of those non-degenerate cases can be elegantly reduced as

$$I : \{\bar{S} \le S < \infty,\ 0 \le t \le T - \bar{J},\ J = 0\},$$
$$II : \{0 \le S \le \bar{S},\ J \le t \le J + T - \bar{J},\ 0 \le J \le \bar{J}\}.$$

Let $V_1(S, t)$ and $V_2(S, t, J)$ denote the option prices in the region I and II, respectively. Based on the definition of the option, the continuity conditions of both the option price and the "Delta" of the option, it can be shown that the option price should satisfy (cf. [3,18])

$$\mathcal{A}_1 \begin{cases} \dfrac{\partial V_1}{\partial t} + \mathbb{L}V_1 = 0, \\ V_1(S, T - \bar{J}) = 0, \\ \lim\limits_{S\to\infty} V_1(S, t) = 0, \\ V_1(\bar{S}, t) = V_2(\bar{S}, t, 0), \end{cases} \qquad \mathcal{A}_2 \begin{cases} \dfrac{\partial V_2}{\partial t} + \dfrac{\partial V_2}{\partial J} + \mathbb{L}V_2 = 0, \\ V_2(S, t, \bar{J}) = C_A(S, t), \\ V_2(0, t, J) = 0, \\ V_2(\bar{S}, t, J) = V_2(\bar{S}, t, 0), \end{cases} \tag{2.2}$$

$$\text{connectivity condition} : \frac{\partial V_1}{\partial S}(\bar{S}, t) = \frac{\partial V_2}{\partial S}(\bar{S}, t, 0), \tag{2.3}$$

where $\mathcal{A}_1$ is defined on $t \in [0, T - \bar{J}],\ S \in [\bar{S}, \infty)$, $\mathcal{A}_2$ is defined on $t \in [J, T - \bar{J} + J], J \in [0, \bar{J}],\ S \in [0, \bar{S}]$, and operator $\mathbb{L} = \frac{\sigma^2 S^2}{2}\frac{\partial^2}{\partial S^2} + (r - D)S\frac{\partial}{\partial S} - rI$, with I being the identity operator. It should be noted that the last equation in $\mathcal{A}_2$ of (2.2) holds only for $0 \le J < \bar{J}$, i.e, the "knock-in" feature has not been triggered.

One can observe that the above PDE systems are quite similar to those governing a European-style Parisian up-and-out call option. However, there are still several key differences. Firstly, it is obvious that the pricing domains of a Parisian down-and-in option are reversed from those of its up-and-out counterpart. Secondly, the knock-in feature makes the "terminal condition", with respect to J, become non-homogeneous in $\mathcal{A}_2$. This is because the option price is equal to that of the embedded American call option, denoted by $C_A(S, t)$, at the time t it is knocked in. Finally, we have the homogeneous boundary condition in $\mathcal{A}_1$ when S approaches infinity because in this case the option is never knocked in.

Albeit different, the above coupled PDE systems can be solved by adopting the "moving window" technique developed in [3]. In the next section, we shall briefly discuss the solution procedure.

3. Solution of the coupled PDE systems

Following the method of [3] with the same notations, the 3-D PDE systems (2.3) can be further simplified to the following two 2-D PDE systems: