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Localization of small perfectly conducting cracks from far-field pattern with unknown frequency

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ABSTRACT

In the inverse scattering problem, it is well-known that subspace migration yields very accurate locations of small perfectly conducting cracks when the applied frequency is known. In contrast, when the applied frequency is unknown, inaccurate locations are identified via subspace migration with wrong frequency data. This phenomenon has been examined experimentally; however, the reason for its occurrence has not been theoretically investigated. In this study, we analyze the mathematical structure of subspace migration with an unknown frequency by establishing a relationship with Bessel functions of order zero of the first kind. The identified structure of subspace migration and the corresponding results of numerical simulations provide reasons for why subspace migration with an unknown frequency yields an inaccurate crack location and ideas for improvement.

1. Introduction

It is well-known that subspace migration is a fast, effective, and stable non-iterative location search algorithm for small perfectly conducting cracks in the inverse scattering problem (see [1–3] for instance). However, the applied-frequency information must be known for successful application. Therefore, many studies assumed that the applied frequency is known and investigated certain properties of single- and multi-frequency subspace migration algorithms (see [4–8] and the references therein).

However, if one has no *a priori* information regarding the applied frequency, subspace migration is inadequate for the detection of unknown targets. Particularly, for the problem of finding the locations of cracks, some information about the location can be examined, but identifying the exact location is still impossible. Unfortunately, this fact has been heuristically examined through the results of numerical simulations; thus, as far as we know, a mathematical analysis of subspace migration is still needed. This provides the motivation for this study to analyze the structure of subspace migration and to develop an algorithm for finding the exact locations of cracks.

In this manuscript, we extend the research in [4,5] on a structural analysis of subspace migration with unknown frequency information. This is based on the fact that the singular vectors associated with the nonzero singular values of a Multi-Static Response (MSR) matrix can be represented by an asymptotic expansion formula in the existence of cracks. Throughout the careful derivation, we identify that the subspace migration imaging functional can be represented by the square of the Bessel function of order zero of the first kind. On the basis of this representation, we investigate the reason the inexact locations of cracks are identified via subspace migration and develop a simple algorithm for finding the exact locations.

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This paper is organized as follows. In Section 2, we survey two-dimensional direct scattering problems, the asymptotic expansion formula in the presence of small perfectly conducting cracks, and subspace migration. In Section 3, we investigate the structure of subspace migration with an unknown applied frequency by establishing a relationship with the Bessel function of order zero of the first kind. Furthermore, we propose an exact location search algorithm by creating a small scatterer. Section 4 presents some numerical simulation results to support our investigation, and Section 5 presents a short conclusion.

2. Preliminaries

In this section, we briefly introduce two-dimensional direct scattering problems in the presence of small, linear, perfectly conducting cracks, the asymptotic expansion formula, and subspace migration.

2.1. Direct scattering problems and the asymptotic expansion formula

First, we consider two-dimensional electromagnetic scattering by M different linear perfectly conducting cracks with the same small length 2ℓ whose center is at $\mathbf{z}_m$, denoted by Γ_m, $m = 1, 2, \ldots, M$, located in the homogeneous space $\mathbb{R}^2$. For the sake of simplicity, we denote Γ to be the collection of Γ_m. In this paper, we assume that the cracks represented by Γ_m are sufficiently separated from each other.

Let $u_{\text{tot}}(\mathbf{x}, \boldsymbol{\theta})$ satisfy the following Helmholtz equation:

$$\begin{cases} \triangle u_{\text{tot}}(\mathbf{x}, \boldsymbol{\theta}) + \omega^2 u_{\text{tot}}(\mathbf{x}, \boldsymbol{\theta}) = 0 & \text{in } \mathbb{R}^2 \setminus \Gamma \\ u_{\text{tot}}(\mathbf{x}, \boldsymbol{\theta}) = 0 & \text{on } \Gamma, \end{cases} \tag{1}$$

where $\omega = 2\pi/\lambda$ denotes the *unknown* angular frequency with a wavelength λ such that $\ell \ll \lambda$. Here, we assume that ω is positive definite, and ω^2 is not an eigenvalue of (1). We denote $u_{\text{inc}}(\mathbf{x}, \boldsymbol{\theta}) = \exp(i\omega\boldsymbol{\theta} \cdot \mathbf{x})$ as the incident plane wave with the direction $\boldsymbol{\theta}$ on the two-dimensional unit circle $\mathbb{S}^1$ and $u_{\text{scat}}(\mathbf{x}, \boldsymbol{\theta})$ as the unknown scattered field, which satisfies the Sommerfeld radiation condition

$$\lim_{|\mathbf{x}| \to \infty} |\mathbf{x}|^{1/2} \left(\frac{\partial u_{\text{scat}}(\mathbf{x}, \boldsymbol{\theta})}{\partial |\mathbf{x}|} - i\omega u_{\text{scat}}(\mathbf{x}, \boldsymbol{\theta}) \right) = 0,$$

uniformly in all directions $\hat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|$.

The far-field pattern $u_\infty(\hat{\mathbf{x}}, \boldsymbol{\theta}; \omega)$ defined on $\mathbb{S}^1$ can be expressed in the form

$$u_{\text{scat}}(\mathbf{x}, \boldsymbol{\theta}) = \frac{e^{i\omega|\mathbf{x}|}}{|\mathbf{x}|^{1/2}} \left\{ u_\infty(\hat{\mathbf{x}}, \boldsymbol{\theta}; \omega) + O\left(\frac{1}{|\mathbf{x}|}\right) \right\},$$

uniformly in all directions $\hat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|$, and $|\mathbf{x}| \longrightarrow +\infty$. Then, on the basis of [9], the far-field pattern can be represented as the following asymptotic expansion formula.

Lemma 2.1 (*Asymptotic Expansion Formula [9]*). *For $0 < \ell < 2$ and $\ell \ll \lambda$, the far-field pattern can be represented as follows:*

$$\begin{aligned} u_\infty(\hat{\mathbf{x}}, \boldsymbol{\theta}; \omega) &= -\frac{2\pi}{\ln(\ell/2)} \sum_{m=1}^{M} u_{\text{inc}}(\mathbf{z}_m, \boldsymbol{\theta}; \omega) \overline{u_{\text{inc}}(\mathbf{z}_m, \hat{\mathbf{x}}; \omega)} + O\left(\frac{1}{|\ln \ell|^2}\right) \\ &= -\frac{2\pi}{\ln(\ell/2)} \sum_{m=1}^{M} \exp\left(i\omega(\boldsymbol{\theta} - \hat{\mathbf{x}}) \cdot \mathbf{z}_m\right) + O\left(\frac{1}{|\ln \ell|^2}\right). \end{aligned} \tag{2}$$

2.2. Introduction to subspace migration

At this point, we apply (2) to explain an imaging technique known as subspace migration. From [1], subspace migration is based on the structure of the singular vectors of the collected MSR matrix:

$$\mathbb{K} = \left[u_\infty(\boldsymbol{\vartheta}_j, \boldsymbol{\theta}_l) \right]_{j,l=1}^{N} = \begin{bmatrix} u_\infty(\boldsymbol{\vartheta}_1, \boldsymbol{\theta}_1) & u_\infty(\boldsymbol{\vartheta}_1, \boldsymbol{\theta}_2) & \cdots & u_\infty(\boldsymbol{\vartheta}_1, \boldsymbol{\theta}_N) \\ u_\infty(\boldsymbol{\vartheta}_2, \boldsymbol{\theta}_1) & u_\infty(\boldsymbol{\vartheta}_2, \boldsymbol{\theta}_2) & \cdots & u_\infty(\boldsymbol{\vartheta}_2, \boldsymbol{\theta}_N) \\ \vdots & \vdots & \ddots & \vdots \\ u_\infty(\boldsymbol{\vartheta}_N, \boldsymbol{\theta}_1) & u_\infty(\boldsymbol{\vartheta}_N, \boldsymbol{\theta}_2) & \cdots & u_\infty(\boldsymbol{\vartheta}_N, \boldsymbol{\theta}_N) \end{bmatrix},$$

where $u_\infty(\boldsymbol{\vartheta}_j, \boldsymbol{\theta}_l)$ is the far-field pattern with the incident direction $\boldsymbol{\theta}_l$ and observation direction $\boldsymbol{\vartheta}_j$ for $j, l = 1, 2, \ldots, N$. For the sake of simplicity, we assume that the incident and observation directions coincide, i.e., $\boldsymbol{\vartheta}_j = -\boldsymbol{\theta}_j$. Then, because the jlth element of the MSR matrix can be represented as

$$u_\infty(-\boldsymbol{\theta}_j, \boldsymbol{\theta}_l) = -\frac{2\pi}{\ln(\ell/2)} \sum_{m=1}^{M} \exp\left(i\omega(\boldsymbol{\theta}_j + \boldsymbol{\theta}_l) \cdot \mathbf{z}_m\right),$$