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On the existence of minimal periodic solutions for a class of second-order Hamiltonian systems ☆

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ABSTRACT

In this paper, we study the existence of minimal periodic solutions for autonomous second-order Hamiltonian systems with even potentials. Some existence results are obtained by using the variational methods.

1. Introduction and main results

Consider the existence of minimal periodic solutions for the following Hamiltonian systems

$$\ddot{u} + V'(u) = 0, \quad \forall u \in \mathbb{R}^N, \tag{1}$$

where $V \in C^1(\mathbb{R}^N, \mathbb{R})$ and $V'(u)$ denotes the gradient of $V(u)$ in u.

As for the existence of minimal periodic solution of systems (1) the pioneer work should trace back to [1]. Using calculus of variations approach, Rabinowitz [1] established the existence of nonconstant prescribed periodic solutions of (1). Moreover, Rabinowitz conjectured that system (1) possesses a nonconstant solution with any prescribed minimal period under his conditions. Since then, many mathematicians began to study the minimal period problem, see [2–27], and references therein. Among all these results, most of them studied various convex Hamiltonian systems. There are only a few papers dealing with the nonconvex case (see [6–15]). Recently, Long [14] applied some ideas of [12,13,28] to the classical Hamiltonian systems without any convexity assumptions, and proved the existence of nonconstant solutions with prescribed minimal period for system (1). In this paper, motivated by [14], we consider the existence of minimal periodic solutions for systems (1). Our main results are the following theorems.

Theorem 1.1. *Fix $T > 0$ and set $\omega = 2\pi/T$. Suppose that V satisfies the following conditions:*

(V_1) $V \in C^1(\mathbb{R}^N, \mathbb{R})$ and it is even, i.e. $V(-x) = V(x)$ for any $x \in \mathbb{R}^N$;

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(V_2)

$$\frac{\omega^2}{2}|x|^2 - V(x) \to +\infty \quad \text{as } |x| \to \infty;$$

(V_3)

$$\lim_{|x|\to 0} \frac{V(x)}{|x|^2} > \frac{\omega^2}{2}.$$

Then, the system (1) possesses an odd nonconstant periodic solution u with minimal period T.

Corollary 1.2. *Suppose that V satisfies (V_1) and the following conditions:*

(V_4)

$$\frac{V(x)}{|x|^2} \to 0 \quad \text{as } |x| \to \infty;$$

(V_5)

$$\frac{V(x)}{|x|^2} \to +\infty \quad \text{as } |x| \to 0.$$

Then, for every $T > 0$, the system (1) possesses an odd nonconstant periodic solution u with minimal period T.

Remark 1.3. The previous corollary has been already stated by Long in [14, Corollary 3] (see also [8] for a related result when V is bounded). On the other hand, Long [14] proved Theorem 1.1 replacing condition (V_2) by a stronger condition that there exist constants a_1 and $0 < a_2 < \omega^2$ such that

$$V(x) \le \frac{a_2}{2}|x|^2 + a_1, \quad \forall x \in \mathbb{R}^N.$$

2. Proof of main results

Let us consider the functional φ on H_T^1 given by

$$\varphi(u) = \frac{1}{2}\int_0^T |\dot{u}(t)|^2 dt - \int_0^T V(u(t))dt$$

for each $u \in H_T^1$, where

$$H_T^1 = \left\{ u : [0,T] \to \mathbb{R}^N \mid \quad u \text{ is absolutely continuous}, u(0) = u(T),\ \dot{u} \in L^2(0,T;\mathbb{R}^N) \right\}$$

is a Hilbert space with the norm defined by

$$\|u\| = \left(\int_0^T |u(t)|^2 dt + \int_0^T |\dot{u}(t)|^2 dt \right)^{\frac{1}{2}}.$$

The functional φ is continuously differentiable and weakly lower semicontinuous on H_T^1. Moreover, we have

$$\langle \varphi'(u), v \rangle = \int_0^T (\dot{u}(t), \dot{v}(t))dt - \int_0^T (V'(u(t)), v(t))dt$$

for all $u, v \in H_T^1$. Let X_T be the subspace of H_T^1 defined by

$$X_T = \{u \in H_T^1 \mid u(T-t) = -u(t) \text{ for } a.e.\ t \in [0,T]\}.$$

Then X_T is a closed subspace of H_T^1 and, therefore, is a Hilbert space. Moreover, $u(0) = u(T) = 0$ for all $u \in X_T$, hence the norm

$$\|u\|_T = \left(\int_0^T |\dot{u}(t)|^2 dt \right)^{1/2}$$

is equivalent to the norm $\|\cdot\|$ on X_T.

Before giving the proof of our main results, we need the following lemma.

Lemma 2.1. *Suppose that (V_1) holds. Then $\varphi \in C^1(X_T, \mathbb{R})$, and $u \in X_T$ is a critical point of φ restricted to X_T if and only if it is an odd C^2-solution of (1).*

Proof. See [8, Lemma 1.2] (see also [14, Proposition 4] for a more general non-autonomous bi-even potential $V(t,x)$). □