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A generalized shift-splitting preconditioner for saddle point problems ☆

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ABSTRACT

For saddle point problems with symmetric positive definite (1, 1)-block, a generalized shift-splitting preconditioner is presented. Theoretical analysis shows the generalized shift-splitting iteration method is unconditionally convergent. Numerical experiments arising from a model Stokes problem are provided to show the effectiveness of the proposed preconditioner.

1. Introduction

A solution of the system of linear equations with the following block 2×2 structure is considered:

$$\begin{bmatrix} A & B^T \\ -B & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}, \tag{1.1}$$

where, $A \in R^{n\times n}$ is symmetric positive definite, $B \in R^{m\times n}$ with $\text{rank}(B) = m < n$, $x, f \in R^n$, and $y, g \in R^m$. B^T denotes the transpose of B. For convenience, we denote

$$\mathscr{A} = \begin{bmatrix} A & B^T \\ -B & 0 \end{bmatrix}.$$

The linear system (1.1) is called the saddle point problem. It is known that the linear system (1.1) is nonsingular when B is full row rank [1,2]. So these assumptions guarantee the existence and uniqueness of the system of linear equations (1.1); see [1,3].

This class of linear systems of the form (1.1) arises in a variety of scientific computing and engineering applications, including computational fluid dynamics [4–6], constrained least squares problems and generalized least squares problems [7–10], mixed finite element of elliptic PDEs, element-free Galerkin method and so forth; see [1,11–16] and the references therein.

In recent years, there has been a surge of interest in the saddle point problem of the form (1.1), and a large number of stationary iterative methods have been proposed. For example, Bai, Golub and Ng studied the Hermitian and skew-Hermitian

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splitting (HSS) iteration methods for non-Hermitian positive definite linear systems [17], and the HSS preconditioner was proposed by Benzi and Golub [3]; Bai et al. presented a class of parameterized inexact Uzawa methods [18], and Jiang et al. studied a local Hermitian and skew-Hermitian splitting (LHSS) iteration method [19], and others; see [20–24,17,25,26]. The linear system (1.1) also can be solved using Krylov subspace methods [1,13]. The Krylov subspace methods are more efficient than the stationary iterative methods in general [14]. However, Krylov subspace methods tend to converge slowly when applied to the saddle point problem (1.1), and good preconditioners are key ingredients for the success of Krylov subspace methods in the application. Fortunately, a variety of preconditioners have been proposed and studied in many papers; see [1,11,27–38] and their references therein.

Recently, Bai et al. in [39] presented a shift-splitting preconditioner for a non-Hermitian positive definite linear system. For the saddle point problem (1.1), Cao et al. in [14] proposed a shift-splitting preconditioner

$$\mathscr{P}_{\mathrm{SS}} = \frac{1}{2}(\alpha I + \mathscr{A}) = \frac{1}{2}\begin{bmatrix} \alpha I + A & B^T \\ -B & \alpha I \end{bmatrix}, \tag{1.2}$$

and a local shift-splitting preconditioner

$$\mathscr{P}_{\mathrm{LSS}} = \frac{1}{2}\begin{bmatrix} A & B^T \\ -B & \alpha I \end{bmatrix}, \tag{1.3}$$

where α is a positive constant and I is an identity matrix (with appropriate dimension). The unconditionally convergent property of the shift-splitting iteration and the spectrum distribution of the preconditioned matrix by the local shift-splitting preconditioner have been studied in detail in [14]. In this paper, this idea is generalized and a generalized shift-splitting preconditioner for the saddle point problem (1.1) is proposed. Besides, the convergence of the generalized shift-splitting iteration method is studied. Numerical experiments of a model Stokes problem are presented to show the effectiveness of the proposed preconditioner.

The remainder of this paper is organized as follows: in Section 2, the generalized shift-splitting preconditioner is described and the convergence properties of the generalized shift-splitting iteration method are studied. In Section 3, numerical experiments are provided to show the feasibility and effectiveness of the generalized method. Finally, some concluding remarks are given in Section 4.

2. The generalized shift-splitting preconditioner

Based on the iteration methods studied in [14], a generalized shift-splitting of the saddle point matrix $\mathscr{A}$ can be constructed as follows:

$$\mathscr{A} = \frac{1}{2}\begin{bmatrix} \alpha I + A & B^T \\ -B & \beta I \end{bmatrix} - \frac{1}{2}\begin{bmatrix} \alpha I - A & -B^T \\ B & \beta I \end{bmatrix}, \tag{2.1}$$

where $\alpha \geq 0$, $\beta > 0$ are two constants and I is the identity matrix (with appropriate dimension). By this special splitting, the following generalized shift-splitting method can be defined for solving the saddle point problem (1.1):

The generalized shift-splitting iteration method: Given an initial guess u^0, for $k = 0, 1, 2, \ldots$, until $\{u^k\}$ converges, compute

$$\frac{1}{2}\begin{bmatrix} \alpha I + A & B^T \\ -B & \beta I \end{bmatrix} u^{k+1} = \frac{1}{2}\begin{bmatrix} \alpha I - A & -B^T \\ B & \beta I \end{bmatrix} u^k + \begin{bmatrix} f \\ g \end{bmatrix}, \tag{2.2}$$

where $\alpha \geq 0$, $\beta > 0$ are two given constants. It is easy to see that the iteration matrix of the generalized shift-splitting iteration is

$$\Gamma = \begin{bmatrix} \alpha I + A & B^T \\ -B & \beta I \end{bmatrix}^{-1} \begin{bmatrix} \alpha I - A & -B^T \\ B & \beta I \end{bmatrix}. \tag{2.3}$$

The splitting preconditioner that corresponds to the generalized shift-splitting iteration (2.2) is given by

$$\mathscr{P}_{\mathrm{GSS}} = \frac{1}{2}\begin{bmatrix} \alpha I + A & B^T \\ -B & \beta I \end{bmatrix}, \tag{2.4}$$

which is called the generalized shift-splitting preconditioner for the saddle point matrix $\mathscr{A}$.

At each step of the generalized shift-splitting iteration or applying the generalized shift-splitting preconditioner $\mathscr{P}_{\mathrm{GSS}}$ within a Krylov subspace method (such as GMRES), a linear system with $\mathscr{P}_{\mathrm{GSS}}$ as the coefficient matrix needs to be solved. That is to say, a linear system of the form

$$\begin{bmatrix} \alpha I + A & B^T \\ -B & \beta I \end{bmatrix} z = r \tag{2.5}$$

needs to be solved for a given vector r at each step. Since the matrix $\mathscr{P}_{\mathrm{GSS}}$ has the following matrix factorization

$$\mathscr{P}_{\mathrm{GSS}} = \frac{1}{2}\begin{bmatrix} I & \frac{1}{\beta}B^T \\ 0 & I \end{bmatrix} \begin{bmatrix} A + \alpha I + \frac{1}{\beta}B^T B & 0 \\ 0 & \beta I \end{bmatrix} \begin{bmatrix} I & 0 \\ -\frac{1}{\beta}B & I \end{bmatrix}. \tag{2.6}$$