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Reachable set bounding for nonlinear perturbed time-delay systems: The smallest bound

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ABSTRACT

In this letter, we propose a new approach to obtain the smallest box which bounds all reachable sets of a class of nonlinear time-delay systems with bounded disturbances. A numerical example is studied to illustrate the obtained result.

1. Introduction

Notations: $\mathbb{R}^n(\mathbb{R}^n_+)$ is n-dimensional (nonnegative) vector space; $e_i = [0_{1\times(i-1)}\ 1\ 0_{1\times(n-i)}]^T \in \mathbb{R}^n$ is ith-unit vector in $\mathbb{R}^n$; for three vectors $x = [x_1\ x_2\ \cdots\ x_n]^T \in \mathbb{R}^n$, $y = [y_1\ y_2\ \cdots\ y_n]^T \in \mathbb{R}^n$ and $q = [q_1\ q_2\ \cdots\ q_n]^T \in \mathbb{R}^n_+$, two $n \times n$-matrices $A = [a_{ij}]$, $B = [b_{ij}]$, the following notations will be used in our development: $|x| := [|x_1|\ |x_2|\ \cdots\ |x_n|]^T$; $x \prec y(\preceq y)$ means that $x_i < y_i(\le y_i)$, $\forall i = 1, \dots, n$; $A \prec B(\preceq B)$ means that $a_{ij} < b_{ij}(\le b_{ij})$, $\forall i, j = 1, \dots, n$; A is nonnegative if $0 \preceq A$; A is *essentially nonnegative* (called a Metzler matrix) if $a_{ij} \ge 0$, $\forall i, j = 1, \dots, n$, $i \ne j$; $\mu(A)$ stands for the spectral abscissa of matrix A; $\mathcal{B}(0, q) = \{x \in \mathbb{R}^n : |x| \preceq q\}$ is a box in $\mathbb{R}^n$.

Reachable set of dynamic systems perturbed by bounded inputs (disturbances) is the set of all the states starting from the origin by inputs with peak value [1–3]. Reachable set bounding of perturbed dynamic systems and its applications are important research areas in control theory and have attracted much attention during the past decades (see, [1–3] and the references therein). Recently, there is a growing interest in the problem of reachable set bounding for perturbed systems with time-delays [1–11] and most of the existing results are only for linear systems. In this letter, we present a new result for a class of nonlinear time-delay systems with bounded disturbances as below:

$$\begin{aligned} &\dot{x}(t) = Ax(t) + F(t, x(t), x(t-\tau_1(t)), \dots, x(t-\tau_m(t), \omega(t))), \quad t \ge 0, \\ &x(s) = 0, \quad s \in [-h, 0] \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $\omega(t) \in \mathbb{R}^1$ is the disturbance vector satisfying

$$|\omega(t)| \le \overline{\omega}, \tag{2}$$

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$\overline{\omega}$ is a given positive scalar, time-varying delays $\tau_0(t) \equiv 0$ and $0 \leq \tau_k(t) \leq \overline{\tau}_k \leq h,\ k = 1, \ldots, m$ are given continuous functions, $\overline{\tau}_k$, $k = 1, \ldots, m$ are nonnegative scalars, $F(t, \ldots) \in \mathbb{R}^n$ is a given continuous function satisfying

$$|F(t, x(t), \ldots, \omega(t))| \leq \sum_{k=0}^{m} A_k|x(t - \tau_k(t))| + B|\omega(t)|, \tag{3}$$

A is an essentially nonnegative matrix and A_k, $k = 0, \ldots, m$, B are nonnegative matrices. Note that there are many classes of systems such as time-varying systems, switched systems [12–14] which can be reformulated into the form of (1), (2), (3).

For linear perturbed time-delay systems whose matrices are convex combinations of constant matrices (polytopic uncertainties), the most widely used approach is based on the Lyapunov–Krasovskii (or Razumikhin) functional method [1–10]. For linear time-delay systems whose matrices are matrix functions, a novel approach which does not involve the Lyapunov–Krasovskii functional method has just been proposed in [11]. So far, there is no available approach which provides the smallest bound of reachable sets of considered time-delay systems. In this letter, inspired by the comparison method proposed in [15], we propose a new and simple approach to obtain *the smallest box* which bounds all reachable sets of the above nonlinear perturbed time-delay system. Lastly, we study the numerical example considered in [11] to illustrate the proposed approach.

2. Main result

For simplicity, we consider system (1) with one delay, i.e. $m = 1$. The obtained result can be extended to the case where system (1) has multiple delays. Let us first consider the following three linear nonnegative time-delay systems:

$$\dot{y}(t) = (A + A_0)y(t) + A_1 y(t - \tau_1(t)) + B|\omega(t)|, \tag{4}$$
$$y(s) = \varphi(s),\ s \in [-h, 0],$$
$$\dot{z}(t) = (A + A_0)z(t) + A_1 z(t - \tau_1(t)) + B\overline{\omega}, \tag{5}$$
$$z(s) = \psi(s),\ s \in [-h, 0],$$
$$\dot{u}(t) = (A + A_0)u(t) + A_1 u(t - \tau_1(t)), \tag{6}$$
$$u(s) = \phi(s),\ s \in [-h, 0].$$

Lemma 1 (*[16,17]*). *The above three linear time-delay systems are nonnegative.*

Proof. This Lemma can be seen as a natural extension of Proposition 3.1 in [16]. □

Lemma 2 (*[18]*). *If a positive scalar α exists such that one of the following conditions hold:*
(i) $\mu(\alpha I_n + (A + A_0) + A_1 e^{\alpha\overline{\tau}_1}) < 0$;
(ii) $\exists p \succ 0 : (\alpha I_n + (A + A_0) + A_1 e^{\alpha\overline{\tau}_1})^T p \preceq 0$;
then system (6) is α-exponentially stable, i.e. there is a positive vector function $\varrho(.)$ such that

$$u_\phi(t) \preceq \varrho(\phi)e^{-\alpha t}, \quad \forall t \geq 0. \tag{7}$$

Lemma 3 (*[15]*). *Let $M \in R^{n\times n}$ be a Metzler matrix. Then the following statements are equivalent*
(i) $\mu(M) < 0$.
(ii) *M is invertible and $M^{-1} \preceq 0$.*

Let us denote a solution with initial condition $y(s) = \varphi(s)$, $s \in [-h, 0]$ of system (4) by $y(t, \varphi)$ and a solution with initial condition $z(s) = \psi(s)$, $s \in [-h, 0]$ of system (5) by $z(t, \psi)$. The following two lemmas are useful for our development:

Lemma 4. *If $\varphi(s) \preceq \psi(s)$, $\forall s \in [-h, 0]$ then we have $y(t, \varphi) \preceq z(t, \psi)$, $\forall t \geq 0$.*

Proof. Denote $e(t) = z(t) - y(t)$, $\varepsilon(t) = \overline{\omega} - |\omega(t)|$ and consider the following system

$$\dot{e}(t) = (A + A_0)e(t) + A_1 e(t - \tau_1(t)) + B\varepsilon(t), \tag{8}$$
$$e(s) = \psi(s) - \varphi(s), \quad s \in [-h, 0].$$

By Lemma 1, we have $e(t, \psi - \varphi) \succeq 0$, $\forall t \geq 0$. This implies that $y(t, \varphi) \preceq z(t, \psi)$, $\forall t \geq 0$. The proof of Lemma 4 is completed. □

Lemma 5. *If $\mu(A + A_0 + A_1) < 0$ then there exist a positive $q \in \mathbb{R}^n_+$, a positive scalar α, a positive vector function $\varrho(.)$ such that*

$$q - \varrho(q)e^{-\alpha t} \preceq z(t, 0) \preceq q, \quad \forall t \geq 0, \tag{9}$$

where $z(t, 0)$ is the solution with initial condition $z(s) = 0$, $s \in [-h, 0]$ of system (5).

Proof. Denote

$$q := -(A + A_0 + A_1)^{-1}B\overline{\omega}. \tag{10}$$