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Oscillation of certain nonlinear fractional partial differential equation with damping term

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ABSTRACT

In this paper, we investigate the oscillatory behavior of solutions of the nonlinear fractional partial differential equation with damping and forced term subject to Robin boundary condition by using differential inequality method as well as integral average method. The main results are illustrated by examples.

1. Introduction

Fractional differential equations occur in many research fields such as in modeling mechanical and electrical properties of real materials, as well as in rheological theory and other physical problems [1–4]. Some aspects of fractional differential equations, such as the existence and uniqueness of solutions to Cauchy type problems, the methods for explicit and numerical solutions, and the stability of solutions have been investigated in some papers, see for example [1,5–7].

Some new developments in the oscillatory behavior of solutions of fractional ordinary differential equations have been reported by authors [8–11]. However very little attention has been paid to the oscillatory behavior of fractional partial differential equations. Recently Prakash et al. [12] established the oscillation of time fractional partial differential equations.

In this paper, we use differential inequality method as well as integral average method to establish the oscillation of the solutions of a nonlinear fractional partial differential equation with damping and forced term of the form

$$D^{\alpha}_{+,t}(r(t)D^{\alpha}_{+,t}u(x,t)) + p(t)D^{\alpha}_{+,t}u(x,t) + q(x,t)f(u(x,t)) = a(t)\Delta u(x,t) + g(x,t), \quad (x,t) \in G \tag{1}$$

with the Robin boundary condition

$$\frac{\partial u(x,t)}{\partial N} + \gamma(x,t)u(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}_+, \tag{2}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with piecewise smooth boundary $\partial\Omega$; $\alpha \in (0,1)$ is a constant; $G = \Omega \times \mathbb{R}_+$, $\mathbb{R}_+ = (0,\infty)$, $D^{\alpha}_{+,t}u$ is the Riemann–Liouville fractional derivative of order α of u with respect to t; Δ is the Laplacian operator; $p(t)$ is the coefficient of the diffusion term; $a(t)$, $q(x,t)$ are continuous functions; the continuous function $g(x,t)$

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is the forced term of the equation; γ is a continuous, nonnegative function on $\partial\Omega \times \mathbb{R}_+$ and $\mathbf{n}$ is the unit exterior normal vector to $\partial\Omega$.

Throughout this paper, we assume that the following conditions hold:

(A_1) $r(t) \in C^\alpha((0,\infty);\mathbb{R}_+)$ and $a(t), p(t) \in C((0,\infty);\mathbb{R}_+)$;
(A_2) $q(x,t) \in C(\overline{G};\mathbb{R}_+)$ and $\min_{x\in\Omega} q(x,t) = Q(t)$;
(A_3) $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $xf(x) > 0$ for all $x \neq 0$.
(A_4) $g \in C(\overline{G};\mathbb{R})$ is a continuous function such that $\int_\Omega g(x,t)dx \leq 0$.

By a solution of (1) we mean a function $u \in C^{2,\alpha}(\overline{\Omega}\times\mathbb{R}^+)$ and $D^\alpha_{+,t}u(x,t) \in C^{1,\alpha}(\overline{\Omega}\times\mathbb{R}^+)$ which satisfies (1) on $\overline{G}$ along with the boundary conditions.

2. Preliminaries

In this section, we give the definitions of fractional derivatives and integrals and introduce some notations which are useful throughout this paper.

The following will be used for our convenience:

$$v(t) = \int_\Omega u(x,t)dx, \qquad \xi = \frac{t^\alpha}{\Gamma(1+\alpha)}, \qquad \widetilde{c}(\xi) = c(t), \qquad \widetilde{r}(\xi) = r(t)$$

$$\widetilde{\sigma}(\xi) = \sigma(t), \qquad \widetilde{Q}(\xi) = Q(t), \qquad \xi_0 = \frac{t_0^\alpha}{\Gamma(1+\alpha)}, \qquad \xi_1 = \frac{t_1^\alpha}{\Gamma(1+\alpha)}, \qquad R(t) = I^\alpha_+\left(\frac{p(t)}{r(t)}\right).$$

Definition 2.1. The Riemann–Liouville fractional partial derivative of order $0 < \alpha < 1$ with respect to t of a function $u(x,t)$ is defined by

$$(D^\alpha_{+,t}u)(x,t) := \frac{\partial}{\partial t}\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\nu)^{-\alpha}u(x,\nu)d\nu \tag{3}$$

provided the right hand side is pointwise defined on $\mathbb{R}_+$ where Γ is the gamma function.

Definition 2.2. The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $y : \mathbb{R}_+ \to \mathbb{R}$ on the half-axis $\mathbb{R}_+$ is defined by

$$(I^\alpha_+ y)(t) := \frac{1}{\Gamma(\alpha)}\int_0^t (t-\nu)^{\alpha-1}y(\nu)d\nu \quad \text{for } t > 0 \tag{4}$$

provided the right hand side is pointwise defined on $\mathbb{R}_+$.

Definition 2.3. The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a function $y : \mathbb{R}_+ \to \mathbb{R}$ on the half-axis $\mathbb{R}_+$ is defined by

$$(D^\alpha_+ y)(t) := \frac{d^{\lceil\alpha\rceil}}{dt^{\lceil\alpha\rceil}}\left(I^{\lceil\alpha\rceil-\alpha}_+ y\right)(t) \quad \text{for } t > 0 \tag{5}$$

provided the right hand side is pointwise defined on $\mathbb{R}_+$ where $\lceil\alpha\rceil$ is the ceiling function of α.

Lemma 2.4 ([12]). *Let*

$$F(t) := \int_0^t (t-\nu)^{-\alpha}y(\nu)d\nu \quad \textit{for } \alpha \in (0,1) \textit{ and } t > 0. \tag{6}$$

Then $F'(t) = \Gamma(1-\alpha)(D^\alpha_+ y)(t)$.

Lemma 2.5. *Consider the differential inequality*

$$D^\alpha_+ v(t) + p(t)v(s) \leq 0, \quad t > s > t_0, \tag{7}$$

where $p(t) \in C(\mathbb{R},[0,\infty))$. *If*

$$\liminf_{t\to\infty}\int_s^t (t-\nu)^{\alpha-1}p(\nu)d\nu \geq \frac{1}{e}, \tag{8}$$

then the inequality (7) has no eventually positive solutions.