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Remarks on the logarithmical regularity criterion of the supercritical surface quasi-geostrophic equation in Morrey spaces

Yan Jia, Bo-Qing Dong *

School of Mathematical Sciences, Anhui University, Hefei 230601, PR China

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ABSTRACT

This paper is concerned with the regularity criterion of the weak solutions of the two-dimensional supercritical surface quasi-geostrophic equation in a critical Morrey space. It is proved that if the weak solution $\theta(x,t)$ of the supercritical quasi-geostrophic equation satisfies the growth condition

$$\int_0^T \frac{\|\nabla\theta(t)\|^r_{\dot{M}_{p,q}}}{1+\ln(e+\|\nabla\theta\|_{L^p})}dt < \infty \quad \text{with } \frac{2}{p}+\frac{\alpha}{r}=\alpha,\ \frac{2}{\alpha}<p<\infty,$$

then the solution $\theta(x,t)$ is regular on $\mathbb{R}^2\times(0,T]$. This improves the earlier results by Xiang (2010).

1. Introduction

In 1994, Constantin, Majda and Tabak [1] introduced a simple model to approximate the atmospheric and oceanic fluid [2] which is governed by the following so-called surface quasi-geostrophic equation

$$\begin{cases} \dfrac{\partial\theta}{\partial t}+u\cdot\nabla\theta+\kappa\Lambda^\alpha\theta=0, & (x,t)\in\mathbb{R}^2\times(0,\infty),\\ \theta(x,0)=\theta_0, & x\in\mathbb{R}^2, \end{cases} \tag{1.1}$$

where $\theta(x,t)$ is an unknown scalar potential temperature. $u(x,t)$ is the velocity field determined by

$$u=-\nabla^\perp(-\Delta)^{-\frac{1}{2}}\theta=-R^\perp\theta=(R_2\theta,-R_1\theta) \tag{1.2}$$

with R_j the two-dimensional Riesz transforms.

When $\alpha = 1$, the system (1.1) shares many similar features with three-dimensional incompressible Navier–Stokes equations [2]. Therefore, $\alpha = 1$ is referred to as the critical case, while the supercritical case is $0 < \alpha < 1$ and subcritical case is $1 < \alpha \le 2$.

Much attention has been paid on the existence and regularity of the quasi-geostrophic equation (see [3–5]). In the subcritical case Constantin and Wu [6] successfully proved the global existence for smooth solutions of surface quasi-geostrophic equation (1.1) with smooth initial data. When $\alpha = 1$, the existence for global smooth solution of Eq. (1.1) has

* Corresponding author. Tel.: +86 551 63861306.
E-mail addresses: jyan70@163.com (Y. Jia), bqdong@ahu.edu.cn (B. Dong).

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been obtained by Kiselev, Nazarov and Volberg [7] (see also Caffarelli and Vasseur [3]). In the supercritical case $0 < \alpha < 1$, however, the hard problem on the global existence of smooth solutions remains unsolved.

On the one hand, since under the scaling transformation

$$\theta_\lambda(x,t) = \lambda^{\alpha-1}\theta(\lambda x, \lambda^\alpha t), \quad \forall\, \lambda > 0,$$

if $\theta(x,t)$ is a solution of the supercritical dissipative quasi-geostrophic equation (1.1), so does $\theta_\lambda(x,t)$. The scaling invariant property allows to construct a unique regular solution in the supercritical case under the small initial data (see [8–12] and references therein).

On the other hand, it is important that the regularity of weak solutions of surface quasi-geostrophic equation can be proved by imposing some additional condition on the temperature function θ. Constantin, Majda and Tabak [1] first showed the absence of singularities of the solution θ of the supercritical quasi-geostrophic equation (1.1) if

$$\int_0^T \|\nabla^\perp\theta(t)\|_{L^\infty} dt < \infty. \tag{1.3}$$

Chae [13] further proved the regularity criterion of the supercritical quasi-geostrophic equation (1.1) under the assumption

$$\int_0^T \|\nabla^\perp\theta(t)\|^r_{L^p(\mathbb{R}^2)} dt < \infty \tag{1.4}$$

with

$$\frac{2}{p} + \frac{\alpha}{r} \le \alpha, \quad \frac{2}{\alpha} < p < \infty.$$

Recently, Dong [14] (see also Dong and Chen [15]) studied the regularity criterion of the weak solutions if the solution satisfies

$$\int_0^T \|\nabla^\perp\theta(t)\|^r_{B^s_{p,\infty}} dt < \infty \quad \text{with } \frac{2}{p} + \frac{\alpha}{r} = \alpha + s - 1. \tag{1.5}$$

It should be mentioned that Xiang [16] investigated an interesting logarithmical regularity criteria for weak solutions of the supercritical surface quasi-geostrophic equation in Lebesgue space and obtained the regularity of weak solutions subject to the assumption

$$\int_0^T \frac{\|\nabla\theta\|^r_{L^p}}{1 + \ln(e + \|\nabla\theta\|_{L^\infty})} dt < \infty \tag{1.6}$$

with

$$\frac{2}{p} + \frac{\alpha}{r} \le \alpha, \quad \frac{2}{\alpha} < p < \infty.$$

The objective of this study is to improve the above logarithmical regularity criterion (1.6) from Lebesgue space framework to critical Morrey space framework (see its definition in the next section). More precisely, we will show the regularity of weak solution of the supercritical quasi-geostrophic equation (1.1) when the temperature function satisfies the following growth condition in Morrey space

$$\int_0^T \frac{\|\nabla\theta(t)\|^r_{\dot{M}_{p,q}}}{1 + \ln(e + \|\nabla\theta\|_{L^p})} dt < \infty$$

with

$$\frac{2}{p} + \frac{\alpha}{r} = \alpha, \quad \frac{2}{\alpha} < p < \infty.$$

Our main observation is based on the Hölder type inequality in Morrey space together with the rigorous analysis.

2. Preliminaries and main result

Throughout this paper, C stands for a generic positive constant which may vary from line to line. $L^p(\mathbb{R}^2)$ with $1 \le p \le \infty$ and $\dot{H}^s(\mathbb{R}^2)$ with $s \in \mathbb{R}$ denote the usual Lebesgue space and the homogeneous fractional Sobolev space, respectively.

$$\|f\|_{\dot{H}^s} = \left(\int_{\mathbb{R}^2} |\xi|^{2s} |\hat{f}|^2 d\xi\right)^{1/2}.$$