Applied Mathematics Letters 43 (2015) 108–113

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

A numerical technique for variable fractional functional boundary value problems

Xiuying Li [a], Boying Wu [b,*]

[a] Department of Mathematics, Changshu Institute of Technology, Suzhou, Jiangsu 215500, China

[b] Department of Mathematics, Harbin Institute of Technology, Harbin, Heilongjiang 150001, China

ARTICLE INFO

Article history:
Received 18 October 2014
Received in revised form 15 December 2014
Accepted 15 December 2014
Available online 26 December 2014

Keywords:
Reproducing kernel method
Variable fractional order
Boundary value problems

ABSTRACT

This paper is concerned with an efficient numerical method for solving variable fractional functional boundary value problems. The method is based on the reproducing kernel method. Error estimate is also discussed. Numerical examples are provided to show the accuracy and effectiveness.

1. Introduction

Fractional calculus is a generalization of the ordinary differentiation and integration to arbitrary non-integer order. The main advantage of fractional differential equations is that it can provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. Recently several papers have been devoted to the study of constant fractional differential equations. Variable fractional derivatives are a more recent area. The discussion on variable fractional differential equations is rare. Variable fractional differential equations are still useful in noise reduction and signal processing [1–4], the processing of geographical data [5] and signature verification [6]. Recently, Razminia, Dizaji and Majd [7] discussed the existence of the solution for a generalized fractional differential equation with non-autonomous variable order operators. Sun and Chen [8–10] analyzed some applications of variable fractional derivative. On the other hand, the numerical solution of variable fractional differential equation has been considered in some papers [11–21]. Liu, Shen, Zhang et al. [11–20] proposed some numerical methods for variable fractional partial differential equations based on finite difference method. Zhao, Sun and Karniadakis [21], derived two second-order approximation formulas for the variable-order fractional time derivatives. Yu and Ertürk [22] applied a finite difference method to variable order fractional integro-differential equations.

To the best of our knowledge, limited work has been done in the study on space variable fractional differential equations. In this paper, we shall consider the numerical solutions of the following variable fractional functional boundary value problems, based on the reproducing kernel method (RKM)

$$\begin{cases} D^{\alpha(x)}u + a(x)u'(x) + b(x)u(\tau(x)) = f(x), \quad 0 \le x \le 1, \\ u(0) = \mu_1, \qquad u(1) = \mu_2 \end{cases} \tag{1.1}$$

* Corresponding author. Tel.: +86 045186402408; fax: +86 045186415647.
E-mail addresses: lixiuying1112@126.com (X.Y. Li), mathwby@sina.com (B.Y. Wu).

http://dx.doi.org/10.1016/j.aml.2014.12.012
0893-9659/

where $D^{\alpha(x)}$ denotes the variable order Caputo fractional derivative, $1 < \alpha(x) \le 2$, μ_1, μ_2 are constants, $\tau(x) \in C^1[0,1]$, $a(x), b(x) \in W^1[0,1]$, $u(x) \in W^4[0,1]$. Note here that $\tau(x)$ may be larger or smaller than x. $D^{\alpha(x)}$ is defined by

$$D^{\alpha(x)}u(x) = \frac{1}{\Gamma(2-\alpha(x))}\int_0^x (x-t)^{1-\alpha(x)}u''(t)dt. \tag{1.2}$$

2. Numerical method for (1.1)

To solve (1.1) using the RKM, firstly, we homogenize the boundary conditions. Introducing a new unknown function

$$v(x) = u(x) - \phi(x)$$

where $\phi(x) = \gamma_0 + \gamma_1 x$.

Problem (1.1) can be equivalently reduced to the following problem

$$\begin{cases} D^{\alpha(x)}v + a(x)v'(x) + b(x)v(\tau(x)) = g(x), & 0 \le x \le 1, \\ v(0) = 0, \quad v(1) = 0, \end{cases} \tag{2.1}$$

where $g(x) = f(x) - (a(x)\phi'(x) + b(x)\phi(\tau(x)))$.

Since $v(x) \in W^4[0,1]$, the use of integration by part gives

$$\begin{cases} \dfrac{x^{2-\alpha(x)}}{\Gamma(3-\alpha(x))}v''(0) + \dfrac{1}{\Gamma(3-\alpha(x))}\displaystyle\int_0^x (x-s)^{2-\alpha(x)}v'''(s)ds + a(x)v'(x) + b(x)v(\tau(x)) = g(x), \\ v(0) = 0, \quad v(1) = 0. \end{cases} \tag{2.2}$$

It should be noted that $v''(0)$ in (2.2) is unknown. This may be result in great difficulty in finding numerical methods for solving (2.2). However, this is not an impediment to developing our method for (2.2) based on the reproducing kernel theory. Here we put the unknown term into a linear operator defined by

$$Lv(x) = \frac{x^{2-\alpha}}{\Gamma(3-\alpha(x))}v''(0) + \frac{1}{\Gamma(3-\alpha(x))}\int_0^x (x-s)^{2-\alpha(x)}v'''(s)ds + a(x)v'(x) + b(x)v(\tau(x)).$$

To solve (2.2), first we define some reproducing kernel spaces which will be used later.

Definition 2.1. Reproducing kernel space $W^4[0,1] = \{u(x) \mid u'''(x)$ is absolutely continuous, $u^{(4)}(x) \in L^2[0,1], u(0) = 0, u(1) = 0\}$, equipped with the following inner product and norm

$$(u(y), v(y))_4 = u(0)v(0) + u'(0)v'(0) + u(1)v(1) + u'(1)v'(1) + \int_0^1 u^{(4)}v^{(4)}dy$$

and

$$\|u\|_4 = \sqrt{(u,u)_4}.$$

By using the method in [23], the reproducing kernel of $W^4[0,1]$ can be obtained by

$$k(x,y) = \begin{cases} k_1(x,y), & y \le x, \\ k_1(y,x), & y > x, \end{cases} \tag{2.3}$$

where $k_1(x,y) = -\frac{1}{5040}[(x-1)y(x^6y(2y-3) + x^5(11-5y)y - 5x^4y(y+2) + 10x^3y(3y-1) + x^2(2y^6 - 7y^5 - 10102y^2 + 15132y - 5040) + x(-y^6 + 7y^5 + 5040y^2 - 10080y + 5040) - y^6)]$.

Definition 2.2. Reproducing kernel space $W^1[0,1] = \{u(x) \mid u(x)$ is absolutely continuous, $u'(x) \in L^2[0,1]\}$, with the following inner product and norm

$$(u(y), v(y))_1 = u(0)v(0) + \int_0^1 u'v'dy$$

and

$$\|u\|_1 = \sqrt{(u,u)_1}, \quad u, v \in W^1[0,1].$$

Similarly, we can obtain its reproducing kernel

$$\bar{k}(x,y) = \begin{cases} 1+y, & y \le x, \\ 1+x, & y > x. \end{cases} \tag{2.4}$$