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Finite spectrum of 2nth order boundary value problems[☆]

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ABSTRACT

For any even positive integer $2n$ and any positive integer m we construct a class of regular self-adjoint and non-self-adjoint boundary value problems whose spectrum consists of at most $(2n-1)m+1$ eigenvalues. Our main result reduces to previously known results for the cases $n=1$ and $n=2$. In the self-adjoint case with separated boundary conditions this upper bound can be improved to $n(m+1)$.

1. Introduction

We study boundary value problems consisting of the equation

$$(py^{(n)})^{(n)} + qy = \lambda wy, \quad \text{on } J = (a, b), \ -\infty < a < b < +\infty \tag{1.1}$$

with boundary conditions

$$AY(a) + BY(b) = 0, \quad A, B \in M_{2n}(\mathbb{C}). \tag{1.2}$$

Here and below $M_{k,m}(F)$ denotes the $k \times m$ matrices over $F = \mathbb{C}$, the complex numbers, or $F = \mathbb{R}$ the reals, $M_{k,m}(F) = M_k(F)$ when $m = k$; $Y = [y^{[0]}, y^{[1]}, \dots, y^{[2n-1]}]^T$, where T denotes transpose, λ is the spectral parameter, $n \in \mathbb{N} = \{1, 2, 3, \dots\}$, $y^{[j]}$ denotes the quasi-derivatives: $y^{[j]} = y^{(j)}, j = 0, \dots, n-1, y^{[n+j]} = (py^{(n)})^{(j)}, \ j = 0, \dots, n-1$, where $y^{(j)}$ denotes the ordinary derivative.

The coefficients are assumed to satisfy the minimal condition

$$r = 1/p, q, w \in L(J, \mathbb{C}), \tag{1.3}$$

where $L(J, \mathbb{C})$ is the set of complex valued functions which are Lebesgue integrable on J.

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It is well known that condition (1.3) implies that the quasi-derivatives $y^{[j]}$ have finite limits at the endpoints a, b and thus (1.1), (1.2) is a well defined boundary value problem. Condition (1.3) is minimal in the sense that it is necessary and sufficient for all initial value problems of Eq. (1.1) to have a unique solution on $[a, b]$; see [1].

The next lemma is well known. It defines the characteristic function $\Delta(\lambda)$ which characterizes the eigenvalues of the boundary value problem (1.1), (1.2) as zeros (roots) of $\Delta(\lambda)$.

Lemma 1. *Let (1.3) hold and let $\Phi(x,\lambda) = [\phi_{ij}(x,\lambda)]$, $x \in J$ denote the fundamental matrix of the system representation of Eq. (1.1), (see (3.1)) determined by the initial condition $\Phi(a,\lambda) = I$, $\lambda \in \mathbb{C}$. Then a complex number λ is an eigenvalue of the BVP (1.1), (1.2) if and only if*

$$\Delta(\lambda) = \mathbf{det}[A + B\,\Phi(b,\lambda)] = 0. \tag{1.4}$$

The function $\Delta(\lambda) = \mathbf{det}[A + B\,\Phi(b,\lambda)]$, $\lambda \in \mathbb{C}$ is an entire function and is called the characteristic function of Eq. (1.1) and of its system representation (3.1).

Proof. This is well known. □

It follows from Lemma 1 that – unless $\Delta(\lambda)$ is identically zero – the eigenvalues, if any, of problem (1.1), (1.2) are isolated with no finite accumulation point in $\mathbb{C}$. Thus there are exactly four possibilities:

(1) There are no eigenvalues.
(2) Every complex number is an eigenvalue.
(3) There are a countably infinite number of eigenvalues.
(4) There are n eigenvalues for some $n \in \mathbb{N}$.

Clearly case (2) holds when $A = B = 0$ and case (1) holds when $A = 0$, $B = I$, the identity matrix, since the fundamental matrix $\Phi(b,\lambda)$ is nonsingular for every $\lambda \in \mathbb{C}$. Thus we call cases (1) and (2) degenerate.

Definition 1. We call problem (1.1), (1.2) degenerate if $\Delta(\lambda) \equiv 0$ for all $\lambda \in \mathbb{C}$ or $\Delta(\lambda) \neq 0$ for every $\lambda \in \mathbb{C}$.

In the case when the coefficients satisfy:

$$r = 1/p,\ q,\ w \in L(J,\mathbb{R}), \quad p > 0,\ w > 0 \text{ a.e. on } J, \tag{1.5}$$

and the boundary conditions (1.2) are self-adjoint:

$$AE_{2n}A^* = BE_{2n}B^*, \qquad rank(A : B) = 2n, \tag{1.6}$$

where E_{2n} is the symplectic matrix with $k = 2n$ given by

$$E_k = ((-1)^r \delta_{r,k+1-s})_{r,s=1}^{k}, \quad k = 2, 3, 4, \ldots, \tag{1.7}$$

the boundary value problem (1.1), (1.2) is self-adjoint (has a representation as a self-adjoint operator S in the Hilbert space $H = L^2(J, w)$) and there are a countably infinite number of eigenvalues, i.e., case (3) holds. For $n = 1$ Atkinson [2] proved that the positivity conditions on p and w can be considerably weakened to $r = 1/p \geq 0$, $w \geq 0$ and r and w are both positive on some common subinterval of J. We conjecture that this result of Atkinson holds also for $n > 1$ but this is an open problem.

In this paper we find a class of problems for case (4). This we do by constructing a characteristic function $\Delta(\lambda)$ which is a polynomial in λ. This construction is complicated and involves a partition of the interval J and a construction of nonnegative coefficients r and w which are not positive on any common subinterval of J. Our main results reduce to known results when $n = 1$ [3] and $n = 2$ [4–8]. Our construction has its roots in these papers and also uses an inductive scheme.

The organization of this paper is as follows: Following this Introduction, Section 2 contains the statement of the main results and Section 3 their proofs. These are based on several lemmas, some of which may be of independent interest.

2. Higher order boundary value problems with finite spectrum

In this section we state our main results. For $l = 2m + 1$, $m \in \mathbb{N}$ consider a partition of the interval $J = (a, b)$:

$$a = a_0 < a_1 < a_2 < \cdots < a_{l-1} < a_l = b, \tag{2.1}$$