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A well-balanced scheme able to cope with hydrodynamic limits for linear kinetic models

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ABSTRACT

Well-balanced schemes were introduced to numerically enforce consistency with long-time behavior of the underlying continuous PDE. When applied to linear kinetic models, like the Goldstein–Taylor system, this construction generates discretizations which are inconsistent with the hydrodynamic stiff limit (despite it captures diffusive limits quite well). A numerical hybridization, taking advantage of both time-splitting (TS) and well-balanced (WB) approaches is proposed in order to fix this defect: numerical results show that resulting composite schemes improve rendering of macroscopic fluxes while keeping a correct hydrodynamic stiff limit.

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1. Introduction: loss of consistency in stiff relaxation regime

We are interested in an efficient algorithm for the numerical simulation of the linear system,

$$\partial_t f^{\pm} \pm \partial_x f^{\pm} = \pm\frac{1}{\varepsilon}\left[\left(\frac{1}{2}+\phi(x)\right)f^{-} - \left(\frac{1}{2}-\phi(x)\right)f^{+}\right], \qquad f^{\pm}(t=0,\cdot) = f_0^{\pm}, \tag{1}$$

where $0 \leq f^{\pm}(t,x) \in L^1 \cap L^\infty(\mathbb{R})$ stand for densities of right/left moving particles and $\phi \in C_c^\infty(\mathbb{R})$ is a smooth, position-dependent function such that $\|\phi\|_\infty \leq \frac{1}{2}$. One introduces "macroscopic variables", the density $\rho = f^+ + f^-$ and the flux $J = f^+ - f^-$, which satisfy,

$$\partial_t \rho + \partial_x J = 0, \qquad \partial_t J + \partial_x \rho = \frac{1}{\varepsilon}\big(2\phi(x)\rho - J\big). \tag{2}$$

When the Knudsen number vanishes, $0 < \varepsilon \to 0$, and for well-prepared initial data, the macroscopic density relaxes toward the position-dependent continuity equation, *cf.* [1–3],

$$\partial_t \rho + \partial_x(2\phi(x)\rho) = 0, \qquad \rho(t=0,\cdot) = f_0^+ + f_0^-. \tag{3}$$

Different choices of the function ϕ allow to recover previously studied equations: $\phi \equiv 0$ yields the Goldstein–Taylor model, whereas $\phi(t,x) = \partial_x\varphi(t,x)$, φ the concentration of a chemo-attractant substance corresponds to Greenberg–Alt's model of chemotaxis dynamics (see [4] and references therein).

For both the aforementioned cases, an interesting strategy for deriving reliable numerical approximations lies in following the "well-balanced canvas": given a space-step $\Delta x > 0$, one proceeds by localizing the "collisions" onto a discrete lattice

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on the real line, (see *e.g.* [5,6])

$$\partial_t f^\pm \pm \partial_x f^\pm = \pm \sum_{j\in\mathbb{Z}} \frac{\Delta x}{\varepsilon}\left[\left(\frac{1}{2}+\phi(x)\right)f^- - \left(\frac{1}{2}-\phi(x)\right)f^+\right]\delta(x-x_{j-\frac12}),\quad x_j = j\Delta x.$$

At each abscissa, a "local scattering center" [7] appears and the corresponding Dirac mass induces a discontinuity in $f^\pm$: the jump relation is given by the integral curves of the stationary equations of (1), or equivalently of (2). In the particular case where $\phi\equiv 0$, it suffices to mimic the calculations presented in [8] in order to derive them,

$$2\varepsilon\partial_x\bar f^\pm(x) = \bar f^- - \bar f^+,\qquad \bar f^+(0) = f_L^+,\qquad \bar f^-(\Delta x) = f_R^-.$$

The solutions can be expressed by means of a 2×2 *scattering matrix* S,

$$\begin{pmatrix}\bar f^+(\Delta x)\\ \bar f^-(0)\end{pmatrix} = S\begin{pmatrix}\bar f^+(0)\\ \bar f^-(\Delta x)\end{pmatrix},\qquad S = \frac{1}{1+\Delta x/2\varepsilon}\begin{pmatrix}1 & \Delta x/2\varepsilon\\ \Delta x/2\varepsilon & 1\end{pmatrix},$$

which turns out to be *bi-stochastic*, thus ensuring preservation of both L^∞ and L^1 norms. Using standard notation, $x_j = j\Delta x$, $t^n = n\Delta t$ for $j,n\in\mathbb{Z}\times\mathbb{N}$, we set up numerical approximations $f^\pm_{j,n}\simeq f^\pm(t^n,x_j)$. Accordingly, the well-balanced (WB) Godunov scheme, originally derived in [8], rewrites,

$$\begin{pmatrix}f^+_{j,n+1}\\ f^-_{j-1,n+1}\end{pmatrix} = \left(1-\frac{\Delta t}{\Delta x}\right)\begin{pmatrix}f^+_{j,n}\\ f^-_{j-1,n}\end{pmatrix} + \frac{\Delta t}{\Delta x}\,S^{\Delta x}_{j-\frac12}\begin{pmatrix}f^+_{j-1,n}\\ f^-_{j,n}\end{pmatrix}, \tag{4}$$

where $S^{\Delta x}_{j-\frac12}\equiv S$, $(f^+_{j-1,n},f^-_{j,n})$ is the "incoming state" at $x_{j-\frac12}$, and it comes that:

$$f^\pm_{j,n+1} = f^\pm_{j,n+1} - \frac{\Delta t}{\Delta x}(f^\pm_{j,n} - f^\pm_{j\mp1,n}) \pm \frac{\Delta t}{2\varepsilon+\Delta x}(f^\mp_{j,n} - f^\pm_{j\mp1,n}).$$

Clearly, such a scheme is *inconsistent* with the continuous kinetic equation (1) if $\varepsilon\ll\Delta x$ (however, it is L^1-stable because S is stochastic). This comes from the fact that $\frac{\Delta t}{\varepsilon}\gg\frac{\Delta t}{\varepsilon+\Delta x}$ if the computational grid becomes too coarse with respect to ε, so the relaxation process is severely weakened. Hence the scheme (4) is reliable as long as $\varepsilon\geq O(\Delta x)$, *e.g.* $\varepsilon\geq 2\Delta x$. The next section aims at removing this restriction and restoring overall consistency.

2. A composite scattering/time-splitting (TS) discretization

Hereafter we shall use the shorthand notation, $\mathcal{L}(x;f^\pm) = \left(\frac12+\phi(x)\right)f^- - \left(\frac12-\phi(x)\right)f^+$. Moreover, for $0<\varepsilon$ small enough, we are led to define $\varepsilon_{\rm WB}$, $\varepsilon_{\rm TS}$ in such a manner that

$$\frac{1}{\varepsilon} = \frac{1}{\varepsilon_{\rm WB}} + \frac{1}{\varepsilon_{\rm TS}},\qquad 0<\varepsilon_{\rm WB} = O(\Delta x),\qquad \varepsilon_{\rm TS} = \frac{\varepsilon\cdot\varepsilon_{\rm WB}}{\max(0,\varepsilon_{\rm WB}-\varepsilon)}\in(0,+\infty]. \tag{5}$$

Accordingly, the kinetic model (1) is treated by decomposing the collision terms into:

$$\partial_t f^\pm \mp \frac{\mathcal{L}(x;f^\pm)}{\varepsilon_{\rm TS}} = \mp\partial_x f^\pm \pm \frac{\mathcal{L}(x;f^\pm)}{\varepsilon_{\rm WB}}. \tag{6}$$

The left part of (6), possibly stiff if $\varepsilon\ll 1$, is intended to be handled by time-splitting; the right part, from which any stiffness was extracted thanks to the *ad-hoc* choice of parameters, with (4). Our general strategy is to "correct" the lack of consistency in (4) by modifying the "incoming states" at each interface of the computational grid by means of a conventional time-splitting (TS) algorithm. Hence we proceed by, first, building the WB scheme for (1), and second, by indicating how to amend it in order to restore overall consistency.

- In general, one defines a position-dependent 2×2 scattering matrix $S^{\Delta x}_{j-\frac12}$, $j\in\mathbb{Z}$, by solving a boundary-value problem in the interval $x\in(0,\Delta x)$ for the stationary equations,

$$\varepsilon_{\rm WB}\partial_x\bar f^\pm = \mathcal{L}(x_{j-\frac12};\bar f^\pm),\qquad \bar f^+(0) = f_L^+,\qquad \bar f^-(\Delta x) = f_R^-. \tag{7}$$

Lemma 1. *The scattering matrix $S^{\Delta x}_{j-\frac12}$ associated to (7) reads, for $\phi_{j-\frac12} = \phi(x_{j-\frac12})$,*

$$S^{\Delta x}_{j-\frac12} = \frac{1}{\frac12 + \frac{\phi_{j-\frac12}}{\tanh\left(\phi_{j-\frac12}\frac{\Delta x}{\varepsilon_{\rm WB}}\right)}}\begin{pmatrix}\dfrac{2e}{e-1}\phi_{j-\frac12} & \dfrac12+\phi_{j-\frac12}\\ \dfrac12-\phi_{j-\frac12} & \dfrac{2}{e-1}\phi_{j-\frac12}\end{pmatrix},\qquad e = \exp\left(2\phi_{j-\frac12}\frac{\Delta x}{\varepsilon_{\rm WB}}\right).$$

Being stochastic, it preserves the L^1-norm: $|\bar f^+(\Delta x)| + |\bar f^-(0)| \leq |\bar f^+(0)| + |\bar f^-(\Delta x)|$.

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