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Remark on an improved regularity criterion for the 3D MHD equations

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ABSTRACT

In this paper we investigate the regularity criterion for the local in time classical solution to the 3D incompressible magnetohydrodynamic equations. We prove that if $\nabla \times u$ belongs to $L^2(0, T; \dot{B}^{-1}_{\infty,\infty})$, then the local solution (u, B) can be extended beyond time T. As a consequence, this result extends several previous works.

1. Introduction

In this paper, we are interested in the following incompressible MHD equations (MHD) in dimensions three

$$\begin{cases} \partial_t u + (u \cdot \nabla)u - \mu \Delta u + \nabla \left(\pi + \dfrac{|B|^2}{2} \right) = (B \cdot \nabla)B, \quad (x, t) \in \mathbb{R}^3 \times (0, \infty), \\ \partial_t B + (u \cdot \nabla)B - \nu \Delta B = (B \cdot \nabla)u, \\ \nabla \cdot u = 0, \qquad \nabla \cdot B = 0, \end{cases} \tag{1.1}$$

with the initial condition

$$u(x, 0) = u_0(x), \qquad B(x, 0) = B_0(x), \quad x \in \mathbb{R}^3,$$

where $\mu > 0$ and $\nu > 0$ are real constant parameters. $u = u(x, t) \in \mathbb{R}^3$ denote the velocity, $\pi = \pi(x, t) \in \mathbb{R}$ denotes scalar pressure and $B = B(x, t) \in \mathbb{R}^3$ is the magnetic field, while $u_0(x)$ and $B_0(x)$ are the given initial velocity and initial magnetic field, respectively. The MHD equations govern the dynamics of the velocity and magnetic fields in electrically conducting fluids such as plasmas.

The MHD system (1.1) has been studied extensively and many interesting results have been obtained. G. Duvaut and J.-L. Lions [1] constructed a global Leray–Hopf weak solution and a local strong solution of the 3D incompressible MHD system. M. Sermange and R. Temam [2] further examined the properties of these solutions. But whether this unique local solution can exist globally is an outstanding challenge problem due to the presence of Navier–Stokes equations in the system (1.1). For this reason, there are numerous important progresses on the fundamental issue of the blow-up criteria or regularity criteria to the 3D MHD (see., e.g. [3–12] and references therein).

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Very recently, Benbernou et al. [3] established the following regularity criterion to the system (1.1)

$$\int_0^T (\|\nabla \times u(t)\|^2_{\dot{B}^{-1}_{\infty,\infty}} + \|\nabla \times B(t)\|^2_{\dot{B}^{-1}_{\infty,\infty}})\, dt < \infty.$$

Here we want to state that the velocity field plays a more dominant role than the magnetic field does (see the known results in [8,13]). Motivated by the above cited papers, the purpose of this paper is to establish an improved regularity criterion for local smooth solutions to system (1.1). More precisely, we do not pose any assumption on the magnetic field B. Now the refined regularity criterion in terms of the vorticity $\nabla \times u$ can be stated as follows.

Theorem 1.1. *Assume that $(u_0, B_0) \in H^3(\mathbb{R}^3) \times H^3(\mathbb{R}^3)$ with $\nabla \cdot u_0 = \nabla \cdot B_0 = 0$. Let (u, B) be a local smooth solution of the system (1.1). Suppose that*

$$\int_0^T \|\nabla \times u(t)\|^2_{\dot{B}^{-1}_{\infty,\infty}}\, dt < \infty, \tag{1.2}$$

then the solution (u, B) can be extended past time T. Here $\dot{B}^s_{p,r}$ denotes the homogeneous Besov space.

Remark 1.2. As stated above, Benbernou et al. [3] established the following regularity criterion in terms of both the velocity and magnetic fields

$$\int_0^T (\|\nabla \times u(t)\|^2_{\dot{B}^{-1}_{\infty,\infty}} + \|\nabla \times B(t)\|^2_{\dot{B}^{-1}_{\infty,\infty}}) dt < \infty.$$

Therefore, Theorem 1.1 is a further improvement of the result of work [3].

Remark 1.3. In the case $B = 0$, the MHD reduces to the incompressible Navier–Stokes equations, and what proved in [14] is a straightforward consequence of Theorem 1.1.

Noticing the fact $\|\nabla \times u\|_{\dot{B}^{-1}_{\infty,\infty}} \approx \|u\|_{\dot{B}^{0}_{\infty,\infty}}$, we have the following Corollary

Corollary 1.4. *Assume that $(u_0, B_0) \in H^3(\mathbb{R}^3) \times H^3(\mathbb{R}^3)$ with $\nabla \cdot u_0 = \nabla \cdot B_0 = 0$. Let (u, B) be a local smooth solution of the system (1.1). Suppose that*

$$\int_0^T \|u(t)\|^2_{\dot{B}^{0}_{\infty,\infty}}\, dt < \infty, \tag{1.3}$$

then the solution (u, B) can be extended past time T.

Remark 1.5. Thanks to the well-known embedding BMO $\hookrightarrow \dot{B}^0_{\infty,\infty}$, it is easy to see that Corollary 1.4 is a refined improvement of Theorem 2.1 in [10]. Here BMO denotes the space of functions of bounded mean oscillations.

2. The proof of Theorem 1.1

Proof of Theorem 1.1. This section is devoted to the proof of Theorem 1.1. Before proving the theorem, we first introduce the following conventions and notations which will be used throughout this paper. Throughout the paper, C stands for some real positive constants which may be different in each occurrence. We shall sometimes use the notation $A \lesssim B$ which stands for $A \le CB$. Without loss of generality, we set $\mu = \nu = 1$ in the rest of the paper since their sizes do not play any role in our analysis. The existence and uniqueness of local smooth solutions can be done as in the case of the Euler and Navier–Stokes equations (we refer to [15] for more details), thus we may assume that (u, B) is smooth enough in the interval $[0, T)$. We will establish some a priori bounds that will allow us to extend (u, B) past time T under (1.2). If (1.2) holds, one can deduce that for any small $\epsilon > 0$, there exists $T_0 = T_0(\epsilon) < T$ such that

$$\int_{T_0}^T \|\nabla \times u(t)\|^2_{\dot{B}^{-1}_{\infty,\infty}}\, dt \le \epsilon. \tag{2.1}$$

Therefore, in this section we shall establish the following a priori estimate

$$\limsup_{t\to T^-} (\|\nabla^3 u(t)\|^2_{L^2} + \|\nabla^3 B(t)\|^2_{L^2}) < \infty.$$

First, basic energy estimate tells us

$$\|u(t)\|^2_{L^2} + \|B(t)\|^2_{L^2} + \int_0^t (\|\nabla u\|^2_{L^2} + \|\nabla B\|^2_{L^2})(\tau) d\tau \le C < \infty. \tag{2.2}$$