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On the finite time blow-up for filtration problems with nonlinear reaction

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ABSTRACT

We present results for finite time blow-up for filtration problems with nonlinear reaction under appropriate assumptions on the nonlinearities and the initial data. In particular, we prove first finite time blow-up of solutions subject to sufficiently large initial data provided that the reaction term "overpowers" the nonlinear diffusion in a certain sense. Secondly, under related assumptions on the nonlinearities, we show that initial data above positive stationary state solutions will always lead to finite time blow-up.

1. Introduction

In this paper, we shall study the blow-up of solutions $u = u(x, t)$ to the following filtration problem

$$u_t = \Delta K(u) + \lambda f(u), \qquad x \in \Omega, \quad t > 0, \tag{1a}$$

$$\mathcal{B}(K(u)) \equiv \hat{n} \cdot \nabla K(u) + \beta(x) K(u) = 0, \quad x \in \partial\Omega, \quad t > 0, \tag{1b}$$

$$u(x, 0) = u_0(x) \geq 0, \qquad x \in \Omega, \tag{1c}$$

where Ω is a bounded domain of $\mathbb{R}^N$ with sufficient smooth boundary $\partial\Omega$ and $\hat{n}$ denotes the outer unit normal vector.

In problem (1), the non-linear functions f and K, are supposed to be in $C^3(\mathbb{R})$ and to satisfy the following positivity, growth, monotonicity and convexity assumptions:

$$K(0) \geq 0,\ K(s) > 0, \quad \text{for } s \in \mathbb{R}^+, \quad \text{and} \quad K'(s),\ K''(s) > 0, \quad \text{for } s \in \mathbb{R}_0^+, \tag{2}$$

$$f(s) > 0,\ f'(s) > 0,\ f''(s) > 0, \quad \text{for } \ s \in \mathbb{R}_0^+, \tag{3}$$

$$\int_0^\infty \frac{K'(s)}{f(s)} ds < \infty, \quad \text{which implies} \int_0^\infty \frac{ds}{f(s)} < \infty. \tag{4}$$

Moreover, we assume the constant $\lambda > 0$ to be positive and the coefficient $0 \leq \beta(x) \leq \infty$ to be in $C^{1+\alpha}(\partial\Omega)$ for $\alpha > 0$ wherever it is bounded. Note that $\beta \equiv 0$, $\beta \equiv \infty$ and $0 < \beta < \infty$ specify homogeneous Neumann, Dirichlet and Robin

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boundary conditions, respectively. Moreover, this type of conditions are a consequence of Fourier's law for diffusion when considering either conservation of mass, or conservation of energy (see for example [1]).

We remark that by imposing non-negative initial data $u_0(x) \geq 0$, the assumed positivity $f(u) > 0$ implies the positivity of solutions to (1), i.e. $u > 0$ in Ω for $t > 0$. For the existence of a unique classical local solution $u \in C^{2,1}(\Omega_T)$ to problem (1)–(3) we refer to [1–6] and the references therein.

Under the Osgood type condition (4), which is necessary for the blow-up of solutions, the filtration problem (1) will exhibit a blow-up behaviour if the forcing term is sufficiently "strong" and the initial data are sufficiently "large". This can be illustrated by the following example, which applies Kaplan's method, i.e. it investigates the evolution of the Fourier coefficient

$$B(t) := \int_\Omega u(x,t)\,\phi(x)\,dx, \qquad \dot{B}(t) = \int_\Omega \Delta K(u)\,\phi\,dx + \lambda \int_\Omega f\phi\,dx, \tag{5}$$

where the assumptions on β ensure that $\phi(x)$ can be chosen as the positive (in Ω) and L^1-normalised (i.e. $\|\phi\|_1 = 1$) eigenfunction corresponding to the first eigenvalue μ of the following auxiliary elliptic problem with Robin boundary conditions:

$$\begin{cases} \Delta\phi + \mu\phi = 0, & x \in \Omega, \\ \hat{n}\cdot\nabla\phi + \beta(x)\phi = 0, & x \text{ on } \partial\Omega. \end{cases} \tag{6}$$

After integration by parts in (5) and using the eigenvalue problem (6) we obtain $\dot{B}(t) = -\mu\int_\Omega K\,\phi\,dx + \lambda\int_\Omega f\phi\,dx$. Hence, a comparison condition between f and K of the following kind

$$\forall t > 0, \quad \int_\Omega \big(f(u(x,t)) - K(u(x,t))\big)\,\phi(x)\,dx \geq 0 \tag{7}$$

yields, for $\lambda > \mu$ and by applying Jensen's inequality, $\dot{B}(t) \geq (\lambda-\mu)\int_\Omega f(u(x,t))\,\phi(x)\,dx \geq (\lambda-\mu)f(B)$.

Thus, the Osgood type condition (4) implies the finite-time blow-up of $B(t)$ (cf. [2, Section 4.1]). Moreover, as a by-product of Kaplan's method, the first eigenvalue μ provides a lower bound $\mu < \lambda$, above which blow-up occurs.

These phenomena are connected with the existence of solutions to the steady-state problem corresponding to (1),

$$\begin{cases} \Delta(K(w)) + \lambda f(w) = 0, & x \in \Omega, \\ \mathcal{B}(K(w)) = 0, & x \in \partial\Omega. \end{cases} \tag{8}$$

It has been shown in [2], in the closed spectrum case scenario, that problem (8) exhibits a critical (i.e maximal) value of the parameter λ, say λ^*, such that (any kind of) solution to (8) does not exist for $\lambda > \lambda^*$, while there exist bounded solutions to (8) for all $0 < \lambda \leq \lambda^*$. In fact, there exist at least two solutions to (8) for λ close to λ^*. The case of steady-state solutions to the critical parameter λ^* is more intricate, see [2,3]. Since $\mu \geq \lambda^*$ (see [7,8] or [2] for a proof under the additional assumption (7)), Kaplan's method is in general not sharp enough to treat the full supercritical range $\lambda > \lambda^*$. Nevertheless, in [2, Section 4.2] it has been shown under a mild extra condition on f and K that for all $\lambda > \lambda^*$ blow-up of solutions of (1) subject to any initial data $u_0 \geq 0$ occurs (see also [9–11]).

The aim of this work is to complement the analysis of blow-up of solutions for the above filtration problem (1). In a first result (see Theorem 1), we shall assume generalised comparison conditions and characterise sufficiently large initial data such that the solution to (1) blows up in finite time even in the subcritical region $\lambda < \lambda^*$. Moreover, in Theorem 3, we shall prove that solutions of (1) subject to initial data above the positive solution of the steady state problem (which exists since $\lambda < \lambda^*$) blow up in finite time.

2. Blow-up analysis

In the subcritical case $\lambda < \lambda^*$, the large-time behaviour of solutions to the semi-linear filtration problem (1)–(3) depends strongly on the initial data u_0. In particular, as one would expect, blow-up occurs for large enough initial data. A first result in this direction can again be proven by Kaplan's method. More precisely, we can state the following:

Theorem 1 (*Subcritical Blow-Up for Large Initial Data*). *Let the assumptions (2)–(4) hold. Let μ be the first eigenvalue of the problem (6) and ϕ be the corresponding positive normalised eigenfunction. Let $u(x,t)$ denote the solution of (1). Assume either that there exists a concave function $\psi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ satisfying $\gamma := \limsup_{s\to\infty}\frac{\psi(s)}{s} < \frac{\lambda}{\mu}$ and such that*

$$\int_\Omega \big[\psi(f(u(x,t))) - K(u(x,t))\big]\,\phi(x)\,dx \geq 0, \quad \forall t > 0, \tag{9}$$

or that there exists a convex function $\varphi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ with $\varphi(0) = 0$, $\kappa := \liminf_{r\to\infty}\frac{\varphi(r)}{r} > \frac{\mu}{\lambda}$ such that

$$\int_\Omega \big[f(u(x,t)) - \varphi(K(u(x,t)))\big]\,\phi(x)\,dx \geq 0, \quad \forall t > 0, \text{ and assume that } \int_1^\infty \frac{ds}{K(s)} < \infty. \tag{10}$$

Then, the solution $u(x,t)$ of (1)–(3) blows up in finite time, provided that the initial data $u_0 \geq 0$ are sufficiently large.