Applied Mathematics Letters 42 (2015) 59–63

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Blow-up profiles for positive solutions of nonlocal dispersal equation

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ARTICLE INFO

Article history:
Received 21 October 2014
Accepted 13 November 2014
Available online 27 November 2014

Keywords:
Nonlocal dispersal
Positive solution
Blow-up

ABSTRACT

In this paper, we study the blow-up profiles of the nonlocal dispersal equation. More precisely, we prove that the positive solution of nonlocal dispersal equation has different blow-up profiles, depending on the refuge domain.

1. Introduction and main results

In this paper, we consider the nonlocal dispersal equation

$$\begin{cases} J * u(x) - u(x) = -\lambda u(x) + a(x)u^p(x), & x \in \bar{\Omega}, \\ u(x) = 0, & x \in \mathbb{R}^N \setminus \bar{\Omega}, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$, $p > 1$ and λ is a real parameter. The function J is continuous and

$$D[u](x) = J * u(x) - u(x) = \int_{\mathbb{R}^N} J(x-y)u(y)dy - u(x)$$

denotes a nonlocal dispersal operator. The coefficient $a \in C(\bar{\Omega})$ is nonnegative. Throughout this paper, we make the following assumptions:

(A1) $J \in C(\mathbb{R}^N)$ verifies $J > 0$ in B_1 (the unit ball), $J = 0$ in $\mathbb{R}^N \setminus B_1$, $J(x) = J(-x)$ with $\int_{\mathbb{R}^N} J(x)dx = 1$.
(A2) $a(x) \in C(\bar{\Omega})$, $a(x) \not\equiv 0$ and vanishes in a smooth subdomain Ω_0 of Ω.

Eq. (1.1) has been used to model different diffusion phenomena in the literature and attracted considerable interests, for example, the papers [1–8]. In fact, the principal eigenvalue of the nonlocal equation

$$\begin{cases} J * u - u = -\lambda u, & x \in \bar{\Omega}, \\ u = 0, & x \in \mathbb{R}^N \setminus \bar{\Omega} \end{cases} \tag{1.2}$$

is useful in the study of positive solutions to (1.1). It follows from [9] that (1.2) admits a unique principal eigenvalue $\lambda_P(\Omega)$ associated with a positive eigenfunction and $0 < \lambda_P(\Omega) < 1$. Particularly, the positive solution of (1.1) is well studied, see the work of García-Melián and Rossi [10]. An important result is as follows.

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http://dx.doi.org/10.1016/j.aml.2014.11.009
0893-9659/

Theorem 1.1. *There exists a positive solution u_λ of (1.1) if and only if $\lambda_P(\Omega) < \lambda < \lambda_P(\Omega_0)$. In that case, $u_\lambda \in C(\bar{\Omega})$, it is unique, increasing with respect to λ and verifies*

$$\lim_{\lambda \to \lambda_P(\Omega)+} u_\lambda(x) = 0 \quad \text{uniformly in } \bar{\Omega}$$

and

$$\lim_{\lambda \to \lambda_P(\Omega_0)-} u_\lambda(x) = \infty \quad \text{uniformly in } \bar{\Omega}.$$

In order to reveal the complex influence of heterogeneous environment on the positive solutions of (1.1), in this paper we consider the nonlocal dispersal equation

$$\begin{cases} J * u(x) - u(x) = -\lambda u(x) + [a(x) + \varepsilon] u^p(x), & x \in \bar{\Omega}, \\ u(x) = 0, & x \in \mathbb{R}^N \setminus \bar{\Omega}, \end{cases} \tag{1.3}$$

where $\varepsilon > 0$ is a small perturbation parameter. We know that (1.3) admits a unique positive solution u_ε for every $\lambda > \lambda_P(\Omega)$.

Under the hypotheses of (A1)–(A2), we can prove the following conclusions.

Theorem 1.2. *Let $u_\varepsilon \in C(\bar{\Omega})$ be the positive solution of (1.3) for $\lambda > \lambda_P(\Omega)$.*

(i) *If $\lambda_P(\Omega) < \lambda < \lambda_P(\Omega_0)$, then*

$$\lim_{\varepsilon \to 0} u_\varepsilon(x) = u(x) \quad \text{uniformly in } \bar{\Omega},$$

where u is the unique positive solution of (1.1).

(ii) *If $\lambda > \lambda_P(\Omega_0)$, then*

$$\lim_{\varepsilon \to 0} u_\varepsilon(x) = \infty \quad \text{uniformly in } \bar{\Omega}.$$

The result (ii) of Theorem 1.2 shows that u_ε goes to infinity as $\varepsilon \to 0$. It is natural to ask how the blow-up profiles of u_ε can exist. In the present paper, we will further study the blow-up profiles of u_ε and establish the main results as follows.

Theorem 1.3. *Assume that $u_\varepsilon \in C(\bar{\Omega})$ is the positive solution of (1.3) for $\lambda > \lambda_P(\Omega)$. Let $v_\varepsilon = \varepsilon^{\frac{1}{p-1}} u_\varepsilon$ and $\omega_\varepsilon = \varepsilon^{\frac{1}{p(p-1)}} u_\varepsilon$, we have the following results.*

(i) *If $\lambda_P(\Omega) < \lambda < \lambda_P(\Omega_0)$, then*

$$\lim_{\varepsilon \to 0} v_\varepsilon = \lim_{\varepsilon \to 0} \omega_\varepsilon = 0 \quad \text{uniformly in } \bar{\Omega}.$$

(ii) *If $\lambda > \lambda_P(\Omega_0)$, then*

(ii-a) $\lim_{\varepsilon \to 0} v_\varepsilon(x) = \theta(x)$ *uniformly in $\bar{\Omega}_0$ and*

(ii-b)

$$\lim_{\varepsilon \to 0} \omega_\varepsilon(x) = \left[\frac{\int_{\Omega_0} J(x-y)\theta(y)dy}{a(x)} \right]^{\frac{1}{p}} \quad \text{uniformly in any compact subset of } \bar{\Omega} \setminus \bar{\Omega}_0,$$

where θ satisfies

$$\begin{cases} J * \theta(x) - \theta(x) = -\lambda \theta(x) + \theta^p(x), & x \in \bar{\Omega}_0, \\ \theta(x) > 0, & x \in \bar{\Omega}_0, \\ \theta(x) = 0, & x \in \mathbb{R}^N \setminus \bar{\Omega}_0. \end{cases}$$

Remark 1.4. From Theorem 1.3, we know that the blow-up profiles of positive solutions to (1.3) changes in Ω, depending on the refuge domain Ω_0. On the other hand, (1.3) can be considered as the nonlocal analogue of the local diffusion equation

$$\begin{cases} \Delta v(x) = -\lambda v(x) + [a(x) + \varepsilon] v^p(x), & x \in \Omega, \\ v(x) = 0, & x \in \partial\Omega. \end{cases} \tag{1.4}$$

The structure of positive solutions to (1.4) is well understood [11]. In fact, the positive solution v_ε of (1.4) only blows up in Ω_0 as $\varepsilon \to 0$ and v_ε has the same blow-up profiles in Ω_0. These results are quite different from the nonlocal dispersal problem (1.3).

The rest of this paper is organized as follows. In Section 2, we give some preliminaries and prove Theorem 1.2. In Section 3, we consider the blow-up profile of (1.3).