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A note on the global attractor for weakly damped wave equation

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ABSTRACT

In this short note, we consider the long-time behavior for the solution of weakly damped wave equation with lower regular forcing. The existence of a global attractor is obtained. To verify asymptotic compactness of the semigroup, we present a new method about decomposition of the solution and apply the Strichartz estimate to the wave equation according to the recent progress. Moreover, translational regularity of the attractor is established.

1. Introduction

Let Ω be a bounded smooth domain of $\mathbb{R}^3$. Given $\gamma > 0$, we consider the following weakly damped wave equation:

$$\begin{cases} u_{tt} + \gamma u_t - \Delta u + f(u) = g, & x \in \Omega,\ t > 0, \\ u(x,0) = u_0, \qquad u_t(x,0) = u_1, & x \in \Omega, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \end{cases} \tag{1.1}$$

where $g \in H^{-1}(\Omega)$ is independent of time, $f \in C^1(\mathbb{R})$ and satisfies the following conditions

$$|f'(s)| \le c_1(1 + s^2), \tag{1.2}$$

$$\liminf_{|s|\to\infty} \frac{f(s)}{s} > -\lambda_1, \tag{1.3}$$

where $\lambda_1 > 0$ is the first eigenvalue of $-\Delta$ on Ω with Dirichlet boundary conditions.

Denote $\mathcal{E}^\sigma = H^{1+\sigma}(\Omega) \times H^\sigma(\Omega)$ with $\mathcal{E} = \mathcal{E}^0$ for short, and $\|\cdot\|$ for the norm of $L^2(\Omega)$. It is known that in the case of $g \in L^2(\Omega)$, the asymptotic behavior of problem (1.1) has been investigated by many authors in the last years (see e.g. [1–4]), and the global attractor has been established in the natural phase space under the conditions (1.2) and (1.3). Most recently, in [5], the long-time behavior of the Shatah–Struwe solution of damped quintic wave equation has been considered, the existence of a global attractor has been proved, due to the recent progress in Strichartz estimates for the case of bounded domains (see, e.g. [6]). For the case of the forcing term g only belongs to $H^{-1}(\Omega)$, we refer the reader to [7–9], in which the

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existence and asymptotic regularity of global attractor have been discussed for the strongly damped wave equations, yet, to the best of our knowledge, there is no corresponding results for weakly damped wave equation.

The main aim of this note is to consider long-time behavior for problem (1.1) with $g \in H^{-1}(\Omega)$. As mentioned in [9], since strongly damped wave equation contains the strong damping term $-\Delta u_t$, which brings many advantages to consider the long-time behavior, especially in considering the attractor. However, for the weakly damped wave equation (1.1), it seems difficult to apply the corresponding method to verify asymptotic compactness of the solution semigroup, which is a key point to obtain the existence of attractor. Motivated by [5], in this note, we present a new method about decomposition of the solution to Eq. (1.1) together with the Strichartz estimate to the wave equation (see [6]) to verify asymptotic compactness of the solution semigroup. Furthermore, inspired by [8], translational regularity of the attractor is considered.

Our main results can be stated as follows.

Theorem 1.1. *Let $g \in H^{-1}(\Omega)$, and $f \in C^1(\mathbb{R})$ satisfies (1.2) and (1.3), $\{S(t)\}_{t\geq 0}$ be the semigroup generated by solution of problem (1.1) in $\mathcal{E}$. Then $\{S(t)\}_{t\geq 0}$ possesses a compact global attractor $\mathscr{A}$ in $\mathcal{E}$.*

Remark 1.2. In fact, when $g \in L^2(\Omega)$, the global attractor would be bounded in $(H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$ via simple energy estimates, see [4] and the references therein. However, in the case of $g \in H^{-1}(\Omega)$, we cannot expect any higher regularity of the global attractor than $\mathcal{H}$, due to the solution of the corresponding stationary equation of (1.1) $-\Delta h + f(h) = g$ only belongs to $H_0^1(\Omega)$. Nevertheless, after a translational transform, we can obtain the following regularity result.

Theorem 1.3. *$\mathscr{A} - (h, 0)$ is bounded in $\mathcal{E}^{\frac{1}{3}}$, where h is a solution of the stationary equation of (1.1).*

The above theorems will be proved in Section 4. Before that, in Section 2, we introduce the Strichartz estimate for the linear wave equation which is the key technical tool in our proofs, in Section 3, the well-posedness of problem (1.1) is discussed.

For the notations, denote by C any positive constants which may be different from line to line even in the same line, we also denote the different positive constants by C_i, $i \in \mathbb{N}$, for special differentiation.

2. Preliminary: Strichartz estimate

In this section, we introduce a technical tool which will be used in our proofs, that is, the Strichartz estimate for the following linear damped wave equation:

$$v_{tt} + v_t - \Delta v = G(t), \qquad v(0) = v_0, \qquad v_t(0) = v_1.$$

Proposition 2.1 ([5,6]). *Let $(v_0, v_1) \in \mathcal{E}$, $G \in L^1(0, T; L^2(\Omega))$ and $v(t)$ be a solution of above equation such that $(v, v_t) \in C(0, T; \mathcal{E})$. Then $v \in L^4(0, T; L^{12}(\Omega))$ and the following estimate holds:*

$$\|v\|_{L^4(0,T;L^{12}(\Omega))} \leq C_T(\|\nabla v_0\| + \|v_1\| + \|G\|_{L^1(0,T;L^2(\Omega))}), \tag{2.1}$$

where C_T may depend on T, but is independent of v_0, v_1 and G.

3. Well-posedness

Applying the standard Galerkin method and smooth approximation on g (see, e.g. [10]), we can obtain that problem (1.1) has a unique weak solution. Here, we only state the result as follows.

Lemma 3.1. *For every $T > 0$ and initial data $(u_0, u_1) \in \mathcal{E}$, problem (1.1) admits a unique weak solution $u \in C(0, T; H_0^1(\Omega)) \cap C^1(0, T; L^2(\Omega))$ which satisfies Eq. (1.1) in the sense of distribution, i.e.*

$$-\int_0^T (u_t, \phi_t)dt - \gamma \int_0^T (u, \phi_t)dt + \int_0^T (\nabla u, \nabla\phi)dt + \int_0^T (f(u), \phi)dt = \int_0^T (g, \phi)dt,$$

for any $\phi \in C_0^\infty((0, T) \times \Omega)$, and the mapping $\{u_0, u_1\} \to \{u(t), u_t(t)\}$ is continuous in $\mathcal{E}$. Moreover, the solution u satisfies energy estimate

$$\|u_t(t)\|^2 + \|\nabla u(t)\|^2 + 2\int_\Omega F(u(t)) - 2(g, u(t)) \leq \|u_1\|^2 + \|\nabla u_0\|^2 + 2\int_\Omega F(u_0) - 2(g, u_0), \tag{3.1}$$

where $F(s) = \int_0^s f(\tau)d\tau$.

Therefore, we can define the operator semigroup $\{S(t)\}_{t\geq 0}$ in $\mathcal{E}$ as follows:

$$S(t)\{u_0, u_1\} = \{u(t), u_t(t)\},$$

which is continuous in $\mathcal{E}$.