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Coupled systems of boundary value problems with nonlocal boundary conditions

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ABSTRACT

We consider the coupled system $-x'' = \lambda_1 f(t, y(t))$, $-y'' = \lambda_2 g(t, x(t))$, $t \in (0, 1)$, subject to the coupled boundary conditions $x(0) = H_1(\varphi_1(y))$, $x(1) = 0$ and $y(0) = H_2(\varphi_2(x))$, $y(1) = 0$. Since H_1 and H_2 are nonlinear functions and φ_1 and φ_2 are linear functionals realized as Stieltjes integrals, the boundary conditions may be nonlocal and nonlinear in character. By assuming that φ_1 and φ_2 satisfy a particular decomposition hypothesis together with some growth assumptions on H_1 and H_2 at 0 and $+\infty$, we show that this system can possess at least one positive solution even if no growth conditions are imposed on f and g.

1. Introduction

In this paper we consider the existence of at least one positive solution to the following coupled system of nonlocal boundary value problems (BVPs) with parameters $\lambda_1, \lambda_2 > 0$.

$$\begin{aligned} -x'' &= \lambda_1 f(t, y(t)), \quad t \in (0, 1) \text{ subject to } x(0) = H_1(\varphi_1(y)),\ x(1) = 0 \\ -y'' &= \lambda_2 g(t, x(t)), \quad t \in (0, 1) \text{ subject to } y(0) = H_2(\varphi_2(x)),\ y(1) = 0. \end{aligned} \tag{1.1}$$

In (1.1) the functions $H_1, H_2 : \mathbb{R} \to \mathbb{R}$ are nonlinear functions, whereas the functionals $\varphi_1, \varphi_2 : \mathcal{C}([0, 1]) \to \mathbb{R}$ are linear and can be realized as Stieltjes integrals with signed measures. This latter fact allows us to consider nonlocal terms that are possibly negative even when x or y is nonnegative. Our main result demonstrates that (1.1) can possess at least one positive solution provided that φ_1 and φ_2 satisfy an appropriate collection of relatively mild hypotheses together with the functions H_1 and H_2 possessing superlinear growth at both 0 and $+\infty$. Of note is that we impose no growth conditions on f and g at all. This only works due to the decomposition technique, which we shall briefly describe next.

Roughly speaking, the key fact that we utilize, and which we introduced in [1] and has been further developed subsequently in [2–9], is that if one wishes to consider the nonlinear, nonlocal term $H_1(\varphi_1(y))$ appearing in (1.1) and if one also wishes to impose only asymptotic growth conditions on H_1, then a problem occurs since *a priori* it is unclear what growth condition (especially from below) $\varphi_1(y)$ should satisfy. For instance, if we put

$$\varphi_1(y) := \frac{1}{2} y\left(\frac{1}{2}\right) - \frac{1}{3} y\left(\frac{2}{5}\right), \tag{1.2}$$

then for each $M > 0$, there is $y \in \mathcal{C}([0, 1])$ such that $\varphi_1(y) = 0$ and $\|y\| \geq M$. This presents a problem insofar as utilizing a condition such as, for instance, $\lim_{z \to +\infty} \frac{H_1(z)}{z} = +\infty$. To circumvent this problem, we have noticed that when

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operating within a particular positive cone – see Section 2 in the sequel – functional (1.2) may instead be written in the form $\varphi_1(y) = \varphi_{1,1}(y) + \varphi_{1,2}(y)$, where $\varphi_{1,1}$ essentially traps the negativity of φ_1, whereas $\varphi_{1,2}$ satisfies an appropriate coercivity-like condition of the form $\varphi_{1,2}(y) \geq C_0\|y\|$ for some constant $C_0 > 0$. This then produces the control we need—see Example 2.4 for a concrete illustration of this decomposition.

Finally, to contextualize our result we mention some of the relevant literature. Coupled systems of the sort exemplified by (1.1) equipped with a variety of boundary conditions have been treated by many authors—see the interesting recent paper by Henderson and Luca [10] and the references therein. Furthermore, we note that there has been much research on nonlocal boundary value problems in the past several years. Above and beyond the intrinsic mathematical interest in such problems, they can also be used as models for steady-state heat flow and beam deformation—see, for example, the discussion in [11,12]. In any case, the important work of Infante and Webb [13,14] has provided a very general and effective framework in which to study these sorts of problems. Slightly earlier investigations of Karakostas and Tsamatos [15,16] and of Yang [17,18] also address nonlocal and, in the case of Yang's papers, nonlinear boundary conditions with Stieltjes integral representations. Goodrich [19] then examined the case where nonlinear boundary conditions were asymptotically related to linear functionals. More recently, many investigations by Infante, et al. [11,20–23,12] have addressed nonlocal and possibly nonlinear boundary conditions in a variety of settings such as coupled systems, second- and third-order problems, eigenvalue problems, and existence of multiple positive solutions, as well as the connection of these problems to more general Hammerstein integral equations. A recent paper of Anderson [24] addresses a first-order BVP with nonlocal, nonlinear boundary conditions. Another very recent paper of Karakostas [25] presents an interesting analysis of a broad class of nonlocal boundary value problems, also potentially with nonlinear boundary conditions, and provides several interesting applications. Finally, for the reader interested in the early historical development of nonlocal boundary value problem theory, classical works by Picone [26] and Whyburn [27] may be consulted for their historical value.

However, in all of the previously mentioned investigations there is always some growth imposed on the equivalent of our nonlinearities f and g in (1.1). Consequently, our results demonstrate in the specific context of problem (1.1) that the decomposition technique can be utilized to transfer all growth from f and g to H_1 and H_2 instead. Moreover, we note that due to the coupled nature of the nonlocal, nonlinear boundary conditions, the "unraveling" of these layers of composition requires a degree of care that is not present in the uncoupled case—cf. [10].

2. Main result and discussion

We begin by stating certain of the hypotheses we impose on problem (1.1) as well as the technical preliminaries we require. To this end, we remark that in the sequel the map $(t, s) \mapsto G(t, s)$ defined on the unit square will denote the Green's function for the operator $Ly = -y''$ equipped with Dirichlet boundary conditions. Given a fixed proper subinterval $[a, b]$ of $(0, 1)$, the number $\gamma := \min_{t\in[a,b]}\{t, 1 - t\} = \min\{a, 1 - b\} \in (0, 1)$ satisfies the well-known property $\min_{t\in[a,b]} G(t, s) \geq \gamma G(s, s) = \gamma \max_{t\in[0,1]} G(t, s)$, for each $s \in [0, 1]$.

As mentioned in Section 1 and as stated momentarily, we assume that φ_1 and φ_2 satisfy the decompositions $\varphi_1(y) = \varphi_{1,1}(y) + \varphi_{1,2}(y)$ and $\varphi_2(y) = \varphi_{2,1}(y) + \varphi_{2,2}(y)$ for each $y \in \mathcal{C}([0, 1])$. With this in mind and equipping $\mathcal{C}([0, 1])$ with the usual maximum norm, $\|\cdot\|$, we operate within the cone

$$\mathcal{K} := \left\{ y \in \mathcal{C}([0, 1]) \ : \ y(t) \geq 0, \ \min_{t\in[a,b]} y(t) \geq \gamma\|y\|, \ \varphi_{1,1}(y) \geq 0, \ \varphi_{2,1}(y) \geq 0 \right\},$$

which is a slight modification of the cone introduced by Infante and Webb [13]. Since $\varphi_{1,2}$ and $\varphi_{2,2}$ will be nonnegative on $\mathcal{K}$ by virtue of the coercivity-like assumption in (A0) below, we do not need to include this in $\mathcal{K}$. Finally, the assumptions we make are as follows; note that (A0)–(A2) ensure that the functionals have the proper structure, whereas (A3) is a growth condition imposed on H_1 and H_2. No conditions, other than the continuity and nonnegativity hypotheses of condition (A4), are imposed on either f or g.

(A0) Assume that there are four linear functionals $\varphi_{1,1}, \varphi_{1,2}, \varphi_{2,1}, \varphi_{2,2} : \mathcal{C}([0, 1]) \to \mathbb{R}$ such that $\varphi_1(y) = \varphi_{1,1}(y) + \varphi_{1,2}(y)$ and $\varphi_2(y) = \varphi_{2,1}(y) + \varphi_{2,2}(y)$. Moreover, assume that there exist constants $C_0, D_0 > 0$ such that $\varphi_{1,2}(y) \geq C_0\|y\|$ and $\varphi_{2,2}(y) \geq D_0\|y\|$ for each $y \in \mathcal{K}$. (Note that since both φ_1 and φ_2 are linear, there exist constants C_1 and D_1 such that $|\varphi_1(y)| \leq C_1\|y\|$ and $|\varphi_2(y)| \leq D_1\|y\|$, for each $y \in \mathcal{C}([0, 1])$. Henceforth, C_1 and D_1 shall denote these constants.)

(A1) The functionals described in condition (A0) have the form

$$\varphi_1(y) := \int_{[0,1]} y(t)\, d\alpha_1(t), \ \varphi_{1,1}(y) := \int_{[0,1]} y(t)\, d\alpha_{1,1}(t), \ \varphi_{1,2}(y) := \int_{[0,1]} y(t)\, d\alpha_{1,2}(t),$$

$$\varphi_2(x) := \int_{[0,1]} x(t)\, d\alpha_2(t), \ \varphi_{2,1}(x) := \int_{[0,1]} x(t)\, d\alpha_{2,1}(t), \ \varphi_{2,2}(x) := \int_{[0,1]} x(t)\, d\alpha_{2,2}(t),$$

where each of $\alpha_i, \alpha_{i,j} : [0, 1] \to \mathbb{R}$, $i, j = 1, 2$, is of bounded variation on $[0, 1]$.

(A2) It holds that

$$\int_{[0,1]} G(t, s)\, d\alpha_{1,1}(t), \ \int_{[0,1]} G(t, s)\, d\alpha_{2,1}(t), \ \int_{[0,1]} (1 - t)\, d\alpha_{1,1}(t), \ \int_{[0,1]} (1 - t)\, d\alpha_{2,1}(t) > 0,$$

where each of the first two inequalities holds for every $s \in [0, 1]$.