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On the integrability of a three-dimensional cored galactic Hamiltonian

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ABSTRACT

We characterize when the three-dimensional cored galactic Hamiltonian system with Hamiltonian

$$H = \frac{1}{2}\left(p_x^2 + p_y^2 + \frac{p_z^2}{q}\right) + \sqrt{1 + x^2 + y^2 + \frac{z^2}{q}},$$

is completely meromorphically integrable when $q \in [0.36, 1]$. The key point for this characterization is to transform the non-polynomial cored Hamiltonian system into a polynomial one.

1. Introduction and statement of the main results

We consider the three-dimensional cored galactic Hamiltonian

$$H = \frac{1}{2}\left(p_x^2 + p_y^2 + \frac{p_z^2}{q}\right) + \sqrt{1 + x^2 + y^2 + \frac{z^2}{q}},$$

where $q > 0$. Its associated Hamiltonian system is

$$\begin{aligned}
x' &= p_x, \\
y' &= p_y, \\
z' &= \frac{p_z}{q}, \\
p_x' &= -\frac{x}{\sqrt{1 + x^2 + y^2 + z^2/q}}, \\
p_y' &= -\frac{y}{\sqrt{1 + x^2 + y^2 + z^2/q}}, \\
p_z' &= -\frac{z}{q\sqrt{1 + x^2 + y^2 + z^2/q}},
\end{aligned} \tag{1}$$

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where the prime denotes derivative with respect to the time t. Note that this Hamiltonian system has three degrees of freedom.

The motivation for the choice of the potential

$$\sqrt{1+x^2+y^2+z^2/q}$$

comes from the interest of this potential in galactic dynamics, see for instance [1–12]. The parameter q gives the ellipticity of the potential, which ranges in the interval $0.36 \le q \le 1$. Lower values of q have no physical meaning and greater values of q are equivalent to reverse the role of the coordinate axes. So in this paper we consider $q \in [0.36, 1]$. Note that the parameter q used here is the parameter denoted as q^2 in some other papers where there $q \in [0.6, 1]$.

The main aim of this paper is to study the existence or non-existence of an additional meromorphic first integral F of the 3-dimensional cored galactic Hamiltonian system (1) *independent* of H, i.e. the gradients of F and H are linearly independent at any point of the phase space except perhaps in a zero Lebesgue measure set and such that $\{H, F\} = 0$. The existence of a such second independent and in involution first integral allows to simplify the study of the dynamics in two dimensions. Moreover, we will also study the existence of an additional third meromorphic first integral G of the 3-dimensional cored galactic Hamiltonian system independent with H and F and such that $\{H, G\} = \{F, G\} = 0$. Note that the existence of such two additional analytic first integrals independent and in involution will allow to describe completely the dynamics of a Hamiltonian system with three degrees of freedom, such as the 3-dimensional cored galactic Hamiltonian system (1) (see for more details [13]).

The Hamiltonian vector field X associated to system (1) is

$$X = p_x\frac{\partial}{\partial x} + p_y\frac{\partial}{\partial y} + \frac{p_z}{q}\frac{\partial}{\partial z} - \frac{x}{\sqrt{1+x^2+y^2+z^2/q}}\frac{\partial}{\partial p_x} - \frac{y}{\sqrt{1+x^2+y^2+z^2/q}}\frac{\partial}{\partial p_y} - \frac{z}{q\sqrt{1+x^2+y^2+z^2/q}}\frac{\partial}{\partial p_z}.$$

Let U be an open and dense set in $\mathbb{R}^6$. We say that the non-locally constant function $F: U \to \mathbb{R}$ is a first integral of the vector field X on U, if $F(x(t), y(t), z(t), p_x(t), p_y(t), p_z(t)) =$ constant for all values of t for which the solution $(x(t), y(t), z(t), p_x(t), p_y(t), p_z(t))$ of X is defined in U. Clearly F is a *first integral* of X on U if and only if $XF = 0$ on U. A *meromorphic* first integral is a first integral F being F a meromorphic function. The Hamiltonian H is an analytic first integral of system (1) and thus it is a meromorphic first integral. By definition a Hamiltonian system with 3 degrees of freedom having 3 independent first integrals that are in *involution* is *completely integrable*, see again [13] for more details.

Proposition 1. *When $q = 1$ the 3-dimensional cored galactic Hamiltonian system* (1) *is completely integrable with the first integrals H, $F = yp_x - xp_y$ and $G = zp_x - xp_z$.*

Since $XF = 0$ and $XG = 0$ when $q = 1$, and clearly H, F and G are independent and in involution, the proposition follows. Hence, from now on we will restrict to the case $q \neq 1$, i.e. $q \in [0.36, 1)$.

Note that the 3-dimensional cored galactic Hamiltonian system (1) is not a polynomial differential system, and consequently the Darboux theory of integrability (see for instance [14,15]), which is very useful for finding first integrals, cannot be applied to system (1).

Our main result is the following one.

Theorem 2. *The 3-dimensional cored galactic Hamiltonian system* (1) *with $q \in [0.36, 1)$ is not completely integrable with analytic first integrals.*

The proof of Theorem 2 is given in Section 3.

We must mention that the analytic integrability of the cored galactic Hamiltonian system in the plane, i.e. with two degrees of freedom, has been studied in [16].

2. Meromorphic first integrals of Hamiltonian systems with homogeneous potential

During the last century many integrable natural Hamiltonian systems with Hamilton function of the form

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + V(q_1, \dots, q_n) \tag{2}$$

were found. Inside the class of Hamiltonian systems, the ones with homogeneous polynomial potentials were investigated with a special care. Among lots of results we want to mention the work of Ziglin [17,18] where the author developed an elegant theory which relates the integrability of Hamiltonian systems with properties of the monodromy group of variational equations along a particular solution and formulated the necessary conditions of integrability for complex Hamiltonian systems. Yoshida [19] used this theory to formulate a criterion of the non-existence of an additional first integral for homogeneous Hamiltonian systems. Applications of Ziglin theory are considerably restricted by the assumption about the