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Existence of positive solutions for a nonlocal problem with dependence on the gradient

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ABSTRACT

In this article, we study the existence of positive solutions for the nonlocal problem

$$\begin{cases} -M\left(x, \int_\Omega |\nabla u|^p\, dx\right) \Delta_p u = f(x, u, |\nabla u|^{p-2}\nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $N \geq 3$, $M : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ and $f : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are continuous functions.

1. Introduction

In this article, we are interested in the existence of positive solutions for the nonlocal problem with dependence on the gradient

$$\begin{cases} -M\left(x, \int_\Omega |\nabla u|^p\, dx\right) \Delta_p u = f(x, u, |\nabla u|^{p-2}\nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $N \geq 3$, $1 < p < N$, $M : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying.

(M1) There exist positive constants m_0, m_∞ such that

$$m_0 \leq M(x, t) \leq m_\infty, \quad \forall (x, t) \in \overline{\Omega} \times \mathbb{R};$$

(M2) There exist positive constants R_1 and L_1 such that

$$|M(x, t_1^p) - M(x, t_2^p)| \leq L_1 |t_1 - t_2|^{p-1}, \quad \forall x \in \overline{\Omega} \quad \text{and} \quad |t_1|, |t_2| \leq R_1.$$

Eq. (1.1) is called a nonlocal problem because of the term $M\left(x, \int_\Omega |\nabla u|^p\, dx\right)$, which implies that it is no longer a pointwise identity. This causes some mathematical difficulties which make the study of such a problem particularly interesting. In recent years, nonlocal problems have been studied in many papers, we refer to some interesting papers [1–6]. In this paper, motivated by the ideas introduced by D.G. de Figueiredo et al. [7] and developed by F.J.S.A. Corrêa and G.M. Figueiredo [5,8] we study the existence of positive solutions for problem (1.1). The novelty of this work lies in the fact that f depends

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on the gradient of the solutions. Moreover, it should be noticed that the function M depends on x. This leads (1.1) to a nonlocal and nonvariational problem. In order to overcome the difficulties brought, the technique used here consists of associating with problem (1.1) a family of quasilinear elliptic problems with no dependence on the gradient of the solutions, which is variational, and an iterative scheme.

In order to state the main result we assume that $f : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is continuous satisfying the following hypotheses:

(F0) $f(x, t, |\xi|^{p-2}\xi) = 0$ for all $t \le 0$ and all $x \in \overline{\Omega}, \xi \in \mathbb{R}^N$;

(F1) $\lim_{t\to 0} \frac{f(x,t,|\xi|^{p-2}\xi)}{|t|^{p-1}} = 0$ for all $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^N$;

(F2) There exists $q \in (p, p^*), p^* = \frac{Np}{N-p}$ such that

$$\lim_{t\to+\infty} \frac{f(x, t, |\xi|^{p-2}\xi)}{|t|^{q-1}} = 0 \quad \text{for all } x \in \overline{\Omega} \text{ and } \xi \in \mathbb{R}^N;$$

(F3) There exists $\mu > p$ such that

$$0 < \mu F(x, t, |\xi|^{p-2}\xi) := \mu \int_0^t f(x, s, |\xi|^{p-2}\xi)\, ds \le f(x, t, |\xi|^{p-2}\xi)t$$

for all $t > 0, x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^N$;

(F4) There exist positive constants A_1, A_2 such that

$$F(x, t, |\xi|^{p-2}\xi) \ge A_1 t^\mu - A_2 \quad \text{for all } t > 0,\ x \in \overline{\Omega},\ \xi \in \mathbb{R}^N;$$

(F5) The function $t \mapsto \frac{f(x,t,|\xi|^{p-2}\xi)}{t^{p-1}}$ is increasing in $(0, +\infty)$ for all $x \in \overline{\Omega}$ and all $\xi \in \mathbb{R}^N$;

(F6) There exist positive constants R_2, L_2, L_3 such that

$$|f(x, t_1, |\xi|^{p-2}\xi) - f(x, t_2, |\xi|^{p-2}\xi)| \le L_2|t_1 - t_2|^{p-1}, \quad \forall x \in \overline{\Omega},\ |\xi| \le R_1,\ |t_1|, |t_2| \le R_2$$

and

$$|f(x, t, |\xi_1|^{p-2}\xi_1) - f(x, t, |\xi_2|^{p-2}\xi_2)| \le L_3|\xi_1 - \xi_2|^{p-1}, \quad x \in \overline{\Omega},\ |\xi_1|, |\xi_2| \le R_1,\ |t| \le R_2.$$

Remark 1.1. By (F1) and (F2), given any $\epsilon > 0$, there exists $c_\epsilon > 0$ such that

$$|f(x, t, |\xi|^{p-2}\xi)| \le \epsilon|t|^{p-1} + c_\epsilon|t|^{q-1}, \quad \forall (x, t, \xi) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N. \tag{1.2}$$

Hence, if there exists a constant $K_2 > 0$ such that $|t| \le K_2$ then there exists a constant $C > 0$ depending on K_2 such that

$$\left(\int_\Omega |f(x, t, |\xi|^{p-2}\xi)|^{\frac{p}{p-1}}\, dx\right)^{\frac{p-1}{p}} \le C, \quad \forall \xi \in \mathbb{R}^N. \tag{1.3}$$

Our main result establishes the existence of solutions for problem (1.1) involving the positive number C_p, which appears in the following inequalities in $\mathbb{R}^N$

$$(|x|^{p-2}x - |y|^{p-2}y, x - y) \ge C_p|x - y|^p \quad \text{if } p \ge 2 \tag{1.4}$$

or

$$(|x|^{p-2}x - |y|^{p-2}y, x - y) \ge \frac{C_p|x - y|^2}{(|x| + |y|)^{2-p}} \quad \text{if } 1 < p < 2, \tag{1.5}$$

where $(.,.)$ is the inner product usual in $\mathbb{R}^N$.

Let $W_0^{1,p}(\Omega)$ be the Sobolev space with respect to the norm $\|u\| = \left(\int_\Omega |\nabla u|^p\, dx\right)^{\frac{1}{p}}$. We denote by S_r the best constant in the embedding $W_0^{1,p}(\Omega)$ into the space $L^r(\Omega)$ whose norm is defined by $|u|_r = \left(\int_\Omega |u|^r\, dx\right)^{\frac{1}{r}}$.

Theorem 1.2. *Assume that the conditions* (M1)–(M2) *and* (F0)–(F6) *hold. Moreover, if* $C_p > \frac{L_2}{m_0 S_p^p}$ *and*

$$S_p \left(\frac{m_0 L_3 + L_1 C}{m_0^2 C_p S_p^p - m_0 L_2}\right)^{\frac{1}{p-1}} < 1 \tag{1.6}$$

then problem (1.1) *has a positive solution.*