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Research announcement

Existence of the solution to electromagnetic wave scattering problem for an impedance body of an arbitrary shape

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ABSTRACT

A new proof is given for the existence of the solution to electromagnetic (EM) wave scattering problem for an impedance body of an arbitrary shape. The proof is based on the elliptic systems theory and elliptic estimates for the solutions of such systems.

1. Introduction

Let $D \subset \mathbb{R}^3$ be a bounded domain with a connected smooth boundary S, $D' := \mathbb{R}^3 \setminus D$, $k^2 = const > 0$, $\omega > 0$ is frequency, $\zeta = const$, $Re\,\zeta \geq 0$, be the boundary impedance, $\epsilon > 0$ and $\mu > 0$ are dielectric and magnetic constants, $\epsilon' = \epsilon + i\frac{\sigma}{\omega}$, $\sigma = const \geq 0$, $x \in D'$, $r = |x|$, N is the unit normal to S pointing into D'.

Consider the problem

$$\nabla \times e = i\omega\mu h, \qquad \nabla \times h = -i\omega\epsilon' h \quad \text{in } D', \tag{1}$$

$$r(e_r - ike) = o(1), \quad r \to \infty, \tag{2}$$

$$[N, [e, N]] - \frac{\zeta}{i\omega\mu}[N, curl\, e] = -f. \tag{3}$$

Here f is a given smooth tangential field to S, $[A, B] = A \times B$ is the cross product of two vectors, $A \cdot B$ is their scalar product. Problem (1)–(3), which we call *problem* I, is the scattering problem for electromagnetic (EM) waves for an impedance body D of an arbitrary shape. This problem has been discussed in many papers and books. Uniqueness of its solution has been proved (see, e.g., [1], pp. 81–83). Existence of its solution was discussed much less (see [2], pp. 254–256). Explicit formula for the plane EM wave scattered by a small impedance body ($ka \ll 1$, a is the characteristic size of this body) of an arbitrary shape is derived in [1]. There one can also find a solution to many-body scattering problem in the case of small impedance bodies of an arbitrary shape.

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The aim of this paper is to outline a method for proving the existence of the solution to *problem* I based on elliptic theory and on a result from [3]. It is clear that *problem* I is equivalent to *problem* II, which consists of solving the equation

$$(\Delta + k^2)e = 0 \quad \text{in } D', \tag{4}$$

assuming that e satisfies conditions (2) and (3) and

$$Div\, e = 0 \quad \text{on } S. \tag{5}$$

Conditions (5), (2), and Eq. (4) imply $\nabla \cdot e = 0$ in D'. If *problem* II has a solution e, then the pair $\{e, h\}$ solves *problem* I, provided that $h = curl\, e/(i\omega\mu)$. The solution to *problem* II, if it exists, is unique, because *problem I* has at most one solution and is equivalent to *problem* II. This solution satisfies the following estimate:

$$\|e\|^2 := \|e\|_0 := \int_{D'} |e(x)|^2 w(x)dx \le c, \qquad w(x) := (1 + |x|)^{-d}, \quad d = const > 1. \tag{6}$$

We denote by $H^m(D', w)$ the weighted Sobolev space with the weight w, by $\|e\|_m$ the norm in $H^m(D', w)$, and by $|e|_m$ the norm in $H^m(S)$, where $H^m(S)$ is the usual Sobolev space of the functions on S and m need not be an integer.

Let us outline the ideas of our proof.

Step 1. One checks that problem (5), (3), and Eq. (4) is an elliptic problem, i.e., Eq. (4) is elliptic (this is obvious) and the boundary conditions (5), (3), satisfy the Lopatinsky–Shapiro (LS) condition (see, e.g., [4] for the definition of (LS) condition which is also called ellipticity condition for the operator in (4) and the boundary conditions (5), (3), or the complementary condition, see also [5]).

Step 2. Reduction of *problem* II to the form from which it is clear that *problem* II is of Fredholm type and its index is zero.

Step 3. Derivation of the estimate:

$$|e|_{m+1} \le c|f|_m, \quad m > 1/2, \tag{7}$$

where $Re\, \zeta > 0$, $c = const > 0$ does not depend on e or f.

Let us formulate our result.

Theorem 1. *For any tangential to S field $f \in H^m(S)$ problem II has a (unique) solution $e \in H^{m+(3/2)}(D', w)$, $e|_S \in H^{m+1}(S)$, and estimates (6) and (7) hold.*

In Section 2 we prove Theorem 1.

2. Proof of Theorem 1

Proof of Theorem 1. Step 1. The principal symbol of the operator in (4) is $\xi^2\delta_{pq}$, δ_{pq} is the Kronecker delta, $\xi^2 = \sum_{j=1}^3 \xi_j^2$, so system (4) is elliptic. Let us rewrite (4) and boundary conditions (3) and (5) as follows:

$$P(D)e = (D_1^2 + D_2^2 + D_3^2 - k^2)e = 0 \quad \text{in } D', \tag{8}$$

$$B(D)e := \left\{ \frac{\zeta}{i\omega\mu}[N, curl\, e] - [N, [e, N]] = f, \ \sum_{p=1}^{3} D_p e_p = 0 \right\} \quad \text{on } S, \tag{9}$$

where $D_j = -i\partial/\partial x_j$ and $D = (D_1, D_2, D_3)$. The principal part of (8), which defines its principal symbol, is

$$P'(D) = D_1^2 + D_2^2 + D_3^2, \tag{10}$$

where the prime in $P'(D)$ denotes the principal part of (8). If we take the local coordinate system in which $N = (0, 0, 1)$, then the principal part of the boundary operator (9) is the matrix

$$B'(D) := \frac{\zeta}{i\omega\mu} \begin{pmatrix} -D_3 & 0 & D_1 \\ 0 & -D_3 & D_2 \\ D_1 & D_2 & D_3 \end{pmatrix} \tag{11}$$

and its symbol is

$$B'(\xi) = \frac{\zeta}{i\omega\mu} \begin{pmatrix} -i\xi_3 & 0 & i\xi_1 \\ 0 & -i\xi_3 & i\xi_2 \\ i\xi_1 & i\xi_2 & i\xi_3 \end{pmatrix}. \tag{12}$$

The operator $\frac{\partial}{\partial x_p}$ is mapped onto $i\xi_p$. The principal symbol of the operators in the boundary conditions (3), (5) is calculated in the local coordinates in which x_3-axis is directed along N. The third row in matrix (12) corresponds to condition (5). The first two rows correspond to the expression $[N, curl\, e] := (curl\, e)_\tau$, which is responsible for the principal symbol corresponding to boundary condition (3).