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Positive solutions for a class of singular fractional differential equation with infinite-point boundary value conditions ☆

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ABSTRACT

We consider the following nonlinear fractional differential equation with infinite-point boundary value conditions

$$\begin{cases} D_{0+}^{\alpha}u(t) + q(t)f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ u^{(i)}(1) = \sum_{j=1}^{\infty} \alpha_j u(\xi_j), \end{cases}$$

where $\alpha > 2, n-1 < \alpha \leq n, i \in [1, n-2]$ is a fixed integer, $\alpha_j \geq 0, 0 < \xi_1 < \xi_2 < \cdots < \xi_{j-1} < \xi_j < \cdots < 1\ (j = 1, 2, \ldots)$, $\Delta - \sum_{j=1}^{\infty} \alpha_j \xi_j^{\alpha-1} > 0$, $\Delta = (\alpha-1)(\alpha-2)\cdots(\alpha-i)$. The nonlinear term f permits singularities with respect to both the time and space variables. By introducing height functions of the nonlinear term on some bounded sets and considering integrations of these height functions, several local existence and multiplicity of positive solutions theorems are obtained.

1. Introduction

We consider the existence of positive solutions of the following nonlinear fractional differential equation with infinite-point boundary value conditions

$$\begin{cases} D_{0+}^{\alpha}u(t) + q(t)f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ u^{(i)}(1) = \sum_{j=1}^{\infty} \alpha_j u(\xi_j), \end{cases} \tag{1}$$

where $\alpha > 2, n-1 < \alpha \leq n, i \in [1, n-2]$ is a fixed integer, $\alpha_j \geq 0, 0 < \xi_1 < \xi_2 < \cdots < \xi_{j-1} < \xi_j < \cdots < 1\ (j = 1, 2, \ldots)$, $\Delta - \sum_{j=1}^{\infty} \alpha_j \xi_j^{\alpha-1} > 0$, $\Delta = (\alpha-1)(\alpha-2)\cdots(\alpha-i)$, D_{0+}^{α} is Riemann–Liouville's fractional derivative. The

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nonlinear term f may be singular with respect to both the time and space variables. In this paper, a function $u \in C[0,1]$ is called a positive solution of problem (1) if $u(t) > 0$ on $(0,1)$ and u satisfies (1) on [0,1].

The fractional differential equation has emerged as a new branch in the field of differential equations for their deep backgrounds. For an extensive collection of such results, we refer the readers to the monographs [1–4]. There has been a significant development in nonlocal problems for fractional differential equations or inclusions (see [5–15] and the references therein).

In [5], the author considered the existence results of pseudo-solutions for the following m-point boundary value problem of fractional differential equation in a reflexive Banach space E,

$$\begin{cases} D^{\alpha}x(t)+q(t)f(t,x(t))=0, \quad \text{a.e. on } [0,1], \quad \alpha\in(n-1,n],\ n\geq 2, \\ x(0)=x'(0)=\cdots=x^{(n-2)}(0)=0, x(1)=\displaystyle\sum_{i=1}^{m-2}\zeta_i x(\eta_i). \end{cases} \tag{1a}$$

By means of cone expansion and compression fixed point theorem, Wang and Liu [6] investigated existence of positive solution for the following fractional differential equation in a Banach space E

$$\begin{cases} D_{0+}^{\alpha}u(t)+q(t)f(t,u(t))=0, \quad 0<t<1, \\ u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0, \\ u(1)=\displaystyle\sum_{i=1}^{m}\alpha_i u(\xi_i). \end{cases} \tag{1b}$$

After further discussion on the properties of Green function, under some conditions concerning the first eigenvalue with respect to the relevant linear operator, Wang and Zhang [7] obtained some existence and multiplicity results of positive solutions of BVP (1b). Similar results are extended to more general boundary value problems in [8]. Gao and Han [9] investigated the existence and multiplicity of positive solutions for the following infinite-point fractional boundary value problem

$$\begin{cases} D_{0+}^{\alpha}u(t)+q(t)f(t,u(t))=0, \quad 0<t<1, \\ u(0)=0,\ u(1)=\displaystyle\sum_{i=1}^{\infty}\alpha_i u(\xi_i), \end{cases} \tag{1c}$$

where $1<\alpha\leq 2$ is a real number, $\xi_i\in(0,1)$, $\alpha_i\in[0,+\infty)$ with $\sum_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1}<1$, $q(t)\in C([0,1],[0,+\infty))$, $f(t,u)\in C([0,1]\times[0,+\infty),[0,+\infty))$. Motivated by above papers and Yao's work in [16] and [17], the purpose of this paper is to investigate the local existence of positive solutions for the fractional boundary value problem (1) by introducing suitable height functions.

Compared with (1a)–(1c), BVP (1) has the following two new features. First, it contains ith order for the unknown function. Second, the infinite point is involved in boundary value conditions. The Green function is given and its properties are also discussed in this paper.

2. Preliminaries and several lemmas

Definitions and useful lemmas from the fractional calculus theory can be found in the recent literature [1–3], we omit them here.

Lemma 1. *Given $y\in C[0,1]$, and then the unique solution of the problem*

$$\begin{cases} D_{0+}^{\alpha}u(t)+y(t)=0, \quad 0<t<1, \\ u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0, \\ u^{(i)}(1)=\displaystyle\sum_{j=1}^{\infty}\alpha_j u(\xi_j), \end{cases} \tag{2}$$

can be expressed by

$$u(t)=\int_0^1 G(t,s)y(s)\mathrm{d}s,$$

where

$$G(t,s)=\frac{1}{p(0)\Gamma(\alpha)}\begin{cases} t^{\alpha-1}p(s)(1-s)^{\alpha-1-i}-p(0)(t-s)^{\alpha-1}, & 0\leq s\leq t\leq 1, \\ t^{\alpha-1}p(s)(1-s)^{\alpha-1-i}, & 0\leq t\leq s\leq 1, \end{cases} \tag{3}$$

where $p(s)=\Delta-\sum_{s\leq\xi_j}\alpha_j\left(\frac{\xi_j-s}{1-s}\right)^{\alpha-1}(1-s)^i$. Obviously, $G(t,s)$ is continuous on $[0,1]\times[0,1]$.

Proof. The proof is similar to Lemma 2.2 in [8]; we only need to replace $\Delta-\sum_{j=1}^{m-2}\alpha_j\xi_j^{\alpha-1}$ with $\Delta-\sum_{j=1}^{\infty}\alpha_j\xi_j^{\alpha-1}$. We omit the details here. □