Applied Mathematics Letters 39 (2015) 35–41

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

A new stabilization technique for the nonconforming Crouzeix–Raviart element applied to linear elasticity

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ARTICLE INFO

Article history:
Received 29 April 2014
Received in revised form 8 August 2014
Accepted 9 August 2014
Available online 6 September 2014

Keywords:
Elasticity equations
Mixed finite elements
Crouzeix–Raviart element
Nonconforming method
Korn's inequality

ABSTRACT

We present a stabilized finite element method for a mixed formulation of elasticity equations using the lowest order Crouzeix–Raviart element. The stabilization is done to satisfy Korn's inequality and is based on an oblique projection operator.

1. Introduction

Nonconforming finite elements are becoming more and more popular in approximating the solution of elliptic partial differential equations [1–6]. Recently, nonconforming methods have attracted much research interest [7–10]. In fact, when a mixed formulation is required, a nonconforming approach yields a more flexible approach than a conforming one. Whether it is a nearly incompressible elasticity problem [4–6] or a problem of coupling Stokes with Darcy equations [11–15] or a problem from Reissner–Mindlin plate theory [16–18], a nonconforming method yields an efficient discretization technique.

The lowest order nonconforming Crouzeix–Raviart element is the simplest element which satisfies the *discrete inf–sup* stability condition with the piecewise constant pressure space. Moreover, the basis functions of the nonconforming Crouzeix–Raviart method have smaller support than the basis functions of the standard conforming methods. Since the nonconforming finite element methods relax the strong continuity requirement along element interfaces, the satisfaction of the *discrete inf–sup* condition becomes easier. However, the main hindrance for using the nonconforming approach for elasticity equations and Reissner–Mindlin plate theory is that the lowest order nonconforming approach does not satisfy Korn's inequality in the discrete setting [19–21]. This leads to the loss of coercivity of the associated bilinear form.

If the elasticity problem has only Dirichlet boundary condition on the whole boundary of the domain, the weak formulation of the elasticity problem can be reformulated so that Poincaré inequality can be applied [1,3,6]. In this case, the coercivity holds. However, if Neumann boundary condition is imposed on part of the boundary, this reformulation cannot be applied. In that case, a stabilization term has to be introduced. A stabilization technique is introduced in [5] to deal with the pure traction of the linear elasticity, where the rotation of the displacement is projected onto the space of piecewise constant functions associated with one level coarser mesh. The difficulty here is to compute the projection onto the piecewise constant finite element space on one level coarser mesh. The stabilization used in [21,20] does not use

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http://dx.doi.org/10.1016/j.aml.2014.08.005
0893-9659/

one level coarser mesh but depends on the jump of the displacement across the element interfaces. This technique needs the computation of the jump of displacement across each element interface. This needs a particular data structure and is also quite expensive. Another way of getting Korn's inequality in the two dimensional case is presented in [22], where one component of displacement is discretized by using the standard linear continuous finite element space and the other component by using the nonconforming triangular element. The third approach is not only non-symmetric, but it is also complicated, and its generalization to the three dimensional case would lead to a complicated data structure.

In this paper we introduce a very simple stabilization technique for the Crouzeix–Raviart element to enforce Korn's inequality. The stabilization is based on projecting the displacement field onto the space of the continuous piecewise linear functions. The projection is local as it is defined by means of a biorthogonal system.

The rest of the paper is organized as follows. In the next section we introduce the boundary value problem of linear elasticity and a relevant mixed formulation. We introduce our finite element scheme and show its well-posedness in Section 3 and prove an a priori error estimate in Section 4. Finally, a conclusion is drawn in Section 5.

2. The boundary value problem of linear elasticity

We now introduce the boundary value problem of linear elasticity. We consider a homogeneous isotropic linear elastic material body occupying a bounded domain Ω in $\mathbb{R}^d$, $d \in \{2, 3\}$, with polygonal or polyhedral boundary Γ. For a prescribed body force $\boldsymbol{f} \in [L^2(\Omega)]^d$, the governing equilibrium equation in Ω reads

$$-\operatorname{div}\boldsymbol{\sigma} = \boldsymbol{f}, \tag{1}$$

where $\boldsymbol{\sigma}$ is the symmetric Cauchy stress tensor. The stress tensor $\boldsymbol{\sigma}$ is defined as a function of the displacement $\boldsymbol{u}$ by the Saint-Venant Kirchhoff constitutive law

$$\boldsymbol{\sigma} = \frac{1}{2}\mathcal{C}(\nabla\boldsymbol{u} + [\nabla\boldsymbol{u}]^T), \tag{2}$$

where $\mathcal{C}$ is the fourth-order elasticity tensor. The action of the elasticity tensor $\mathcal{C}$ on a tensor $\boldsymbol{d}$ is defined as

$$\boldsymbol{\sigma} = \mathcal{C}\boldsymbol{d} := \lambda(\operatorname{tr}\boldsymbol{d})\mathbf{1} + 2\mu\,\boldsymbol{d}. \tag{3}$$

Here, $\mathbf{1}$ is the identity tensor, and λ and μ are the Lamé parameters, which are constant in view of the assumption of a homogeneous body, and which are assumed positive. We assume that the displacement satisfies homogeneous Dirichlet boundary condition on $\Gamma_D \subset \Gamma$

$$\boldsymbol{u} = \mathbf{0} \quad \text{on } \Gamma_D, \tag{4}$$

and traction boundary condition is imposed on the rest of the boundary $\Gamma_N := \Gamma \setminus \Gamma_D$

$$\boldsymbol{\sigma}\boldsymbol{n} = \boldsymbol{g}_N \quad \text{on } \Gamma_N, \tag{5}$$

where $\boldsymbol{n}$ is the outer normal vector to the boundary Γ_N, and $\boldsymbol{g}_N \in [L^2(\Gamma_N)]^d$.

2.1. Standard weak formulation

We use standard notations for the Sobolev spaces. Here $L^2(\Omega)$ is the space of square-integrable functions defined on Ω with the inner product and norm being denoted by $(\cdot,\cdot)_{0,\Omega}$ and $\|\cdot\|_{0,\Omega}$, respectively. The space $H^1_D(\Omega)$ consists of functions in $H^1(\Omega)$ which vanish on Γ_D in the sense of traces. The inner product and norm on $H^1(\Omega)$ will be denoted by $(\cdot,\cdot)_{1,\Omega}$ and $\|\cdot\|_{1,\Omega}$, respectively. To write the weak or variational formulation of the boundary value problem, we introduce the space $\boldsymbol{V} := [H^1_D(\Omega)]^d$ of displacements. The inner product and norm on $\boldsymbol{V}$ are inherited from the space $H^1(\Omega)$ in a standard way. We use similar notations for inner products and norms in other Sobolev spaces [23,24].

We define the bilinear form $A(\cdot,\cdot)$ and the linear functional $\ell(\cdot)$ by

$$A : \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R},\; A(\boldsymbol{u},\boldsymbol{v}) := \int_\Omega \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\,d\boldsymbol{x},$$

$$\ell : V \to \mathbb{R}, \qquad \ell(\boldsymbol{v}) := \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,d\boldsymbol{x} + \int_{\Gamma_N} \boldsymbol{g}_N\cdot\boldsymbol{v}\,d\boldsymbol{x}.$$

Then the standard weak form of linear elasticity problem is as follows: given $\ell \in \boldsymbol{V}^*$, find $\boldsymbol{u} \in \boldsymbol{V}$ that satisfies

$$A(\boldsymbol{u},\boldsymbol{v}) = \ell(\boldsymbol{v}), \quad \boldsymbol{v} \in \boldsymbol{V}, \tag{6}$$

where $\boldsymbol{V}^*$ is the dual space of $\boldsymbol{V}$. The assumptions on $\mathcal{C}$ guarantee that $A(\cdot,\cdot)$ is symmetric, continuous, and V-elliptic. Hence by using standard arguments it can be shown that (6) has a unique solution $\boldsymbol{u} \in \boldsymbol{V}$.

2.2. Mixed formulation

We consider a simple mixed formulation of the linear elastic problem using pressure and displacement as two unknown variables. Defining $p = \lambda\operatorname{div}\boldsymbol{u}$ a mixed variational formulation of linear elastic problem (6)is given by: find $(\boldsymbol{u}, p) \in$