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# Nonlinear stability for the three dimensional incompressible flow of nematic liquid crystals

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ABSTRACT

This paper studies the nonlinear stability for the three dimensional incompressible flow of liquid crystals. When the Deborah number $\gamma$ is sufficiently small, we show that the linear stability implies the nonlinear stability in $(\mathbf{L}^p(\mathbf{T}^3), \mathbf{W}^{1,p}(\mathbf{T}^3))$ for all $p \in (1, \infty)$.


## 1. Introduction and main results

We consider the three dimensional incompressible flow of liquid crystals with the external time-independent force

$$\frac{\partial U}{\partial t} + U \cdot \nabla U - \mu \triangle U + \nabla P = -\lambda \mathbf{div}(\nabla d \odot \nabla d) + f_\alpha, \tag{1.1}$$

$$\frac{\partial d}{\partial t} + U \cdot \nabla d = \gamma(\triangle d - |\nabla d|^2 d) + h_\alpha, \tag{1.2}$$

$$\mathbf{div}\, U = 0, \tag{1.3}$$

where $U \in \mathbf{R}^3$ is the velocity, $d \in \mathbf{R}^3$ is the director field for the averaged macroscopic molecular orientations, $P \in \mathbf{R}$ is the pressure arising from the incompressibility, and they all depend on the spatial variable $x \in \mathbf{T}^3$ and the time variable $t$. Positive constants $\mu, \lambda, \gamma$ denote viscosity, the competition between kinetic energy and potential energy, and microscopic elastic relaxation time or the Deborah number for the molecular orientation field, respectively. $f_\alpha$ and $h_\alpha$ denote two external time-independent forces. The symbol $\nabla d \odot \nabla d$ denotes a matrix. It is easy to see that

$$\nabla d \odot \nabla d = (\nabla d)^T \nabla d,$$

$$\mathbf{div}(\nabla d \odot \nabla d) = \nabla \left( \frac{|\nabla d|^2}{2} \right) + (\nabla d)^T \triangle d, \tag{1.4}$$

where $(\nabla d)^T$ denotes the transpose of the $3 \times 3$ matrix $\nabla d$.

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One of the most common liquid crystal phases is the nematic, where the molecules have no positional order, but they have long-range orientational order (see e.g. [1–4]). Since the general Ericksen–Leslie system is very complicated, we only consider a simplified model (1.1)–(1.3) of the Ericksen–Leslie system. It still retains most of the essential features (see e.g. [5–8]). Both the Ericksen–Leslie system and the simplified one (1.1)–(1.3) describe the time evolution of liquid crystal materials under the influence of both the velocity field $u$ and the director field $d$. When system (1.1)–(1.3) driven by the external time-independent force, a natural question is that the nonlinear stability for solutions of system (1.1)–(1.3). If the Deborah number $\gamma > 0$ is sufficiently small, this paper shows that the steady smooth solution $(U_\alpha, d_\alpha)$ is $(\mathbf{L}^q, \mathbf{W}^{1,q})$ nonlinearly stable in the sense of Lyapunov.

Assume that $f_\alpha$ and $h_\alpha$ depend smoothly on the parameter $\alpha_c$. This parameter can be chosen suitably so that $(U_\alpha(x), d_\alpha(x), p_\alpha(x))$ is the smooth solution to the following steady problem

$$U \cdot \nabla U - \mu \triangle U + \nabla P = -\lambda \mathbf{div}(\nabla d \odot \nabla d) + f_\alpha, \tag{1.5}$$

$$U \cdot \nabla d = \gamma(\triangle d - f(d)) + h_\alpha, \tag{1.6}$$

$$\nabla \cdot U = 0, \tag{1.7}$$

$$\lim_{|x|\to\infty} u_\alpha(x) = \mathbf{0}, \qquad \lim_{|x|\to\infty} d_\alpha(x) = \mathbf{0}.$$

We linearize system (1.1)–(1.2) about the steady state $(u_\alpha, d_\alpha, p_\alpha)$ by writing

$$U(x,t) = u(t,x) + U_\alpha(x),$$
$$d(x,t) = z(t,x) + d_\alpha(x),$$
$$p = P - p_\alpha.$$

Then, the deviation $(u, z, p)$ from the stationary $(u_\alpha, d_\alpha, p_\alpha)$ satisfies

$$\frac{\partial u}{\partial t} - \mu\triangle u + u_\alpha \cdot \nabla u + u \cdot \nabla u_\alpha + u \cdot \nabla u + \nabla p$$
$$= -\lambda \nabla\left(\frac{|\nabla z|^2}{2}\right) - \lambda\nabla(|\nabla z||\nabla d_\alpha|) - \lambda(\nabla z)^T \triangle(z + d_\alpha) - \lambda(\nabla d_\alpha)^T \triangle z, \tag{1.8}$$

$$\frac{\partial z}{\partial t} - \gamma\triangle z + u_\alpha \cdot \nabla z + u \cdot \nabla d_\alpha + u \cdot \nabla z = -\gamma|\nabla z|^2 z - \gamma|\nabla z|^2 d_\alpha - \gamma|\nabla d_\alpha|^2 z - 2\gamma|\nabla z||\nabla d_\alpha|(z + d_\alpha), \tag{1.9}$$

where $u = (u_{ij})_{i,j=1,2,3}$ and $\nabla \cdot u = (\sum_{j=1}^3 \partial_{x_1} u_{1j}, \sum_{j=1}^3 \partial_{x_1} u_{2j}, \sum_{j=1}^3 \partial_{x_1} u_{3j})^T$.

We set

$$v = \nabla z, \qquad v_\alpha = \nabla z_\alpha. \tag{1.10}$$

Then we take the gradient of (1.9) and (1.4) to rewrite (1.8)–(1.9) as

$$\frac{\partial u}{\partial t} - \mu\triangle u + u_\alpha \cdot \nabla u + u \cdot \nabla u_\alpha + u \cdot \nabla u + \nabla p$$
$$= -\lambda\nabla\left(\frac{|v|^2}{2}\right) - \lambda\nabla(|v||\nabla d_\alpha|) + \lambda v^T \nabla(v + v_\alpha) + \lambda v_\alpha^T \nabla v, \tag{1.11}$$

$$\frac{\partial v}{\partial t} - \gamma\triangle v + u_\alpha \cdot \nabla v + v\nabla u_\alpha + u \cdot \nabla v_\alpha + v_\alpha \nabla u + u \cdot \nabla v + v \nabla u$$
$$= -\gamma\nabla(|v|^2 z + |v|^2 d_\alpha) - \gamma\nabla(|\nabla d_\alpha|^2 z + 2|v||\nabla d_\alpha|(z + d_\alpha)), \tag{1.12}$$

with the incompressible condition $\nabla \cdot u = 0$. Here we notice that

$$\frac{\partial}{\partial x_k}\left(u_j \frac{\partial d_i}{\partial x_j}\right) = \frac{\partial u_j}{\partial x_k}\frac{\partial d_i}{\partial x_j} + u_j \frac{\partial}{\partial x_j}\left(\frac{\partial d_i}{\partial x_k}\right) = (v\nabla u + u \cdot \nabla v)_{ik}, \quad \forall i, j, k = 1, 2, 3.$$

Note that the incompressible condition gives that $\nabla \cdot (uv^T) = u \cdot \nabla u + u\nabla \cdot u = u \cdot \nabla u$. So we obtain

$$\frac{\partial u}{\partial t} - \mu\triangle u + \nabla \cdot (u_\alpha u^T) + \nabla \cdot (u u_\alpha^T) + \nabla \cdot (uu^T) + \nabla p$$
$$= -\lambda\nabla\left(\frac{|v|^2}{2}\right) - \lambda\nabla(|v||\nabla d_\alpha|) + \lambda v^T \nabla(v + v_\alpha) + \lambda v_\alpha^T \nabla v, \tag{1.13}$$

$$\frac{\partial v}{\partial t} - \gamma\triangle v + u_\alpha \cdot \nabla v + v\nabla u_\alpha + u \cdot \nabla v_\alpha + v_\alpha \nabla u + u \cdot \nabla v + v \nabla u$$
$$= -\gamma\nabla(|v|^2 z + |v|^2 d_\alpha) - \gamma\nabla(|\nabla d_\alpha|^2 z + 2|v||\nabla d_\alpha|(z + d_\alpha)). \tag{1.14}$$

We introduce the definition of Lyapunov stable and unstable about the three dimensional incompressible flow of liquid crystals (see e.g. [9,10]).