Applied Mathematics Letters 39 (2015) 67–72

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Nonoscillatory solutions of higher order differential and delay differential equations with forcing term

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ARTICLE INFO

Article history:
Received 4 July 2014
Received in revised form 16 August 2014
Accepted 16 August 2014
Available online 28 August 2014

Keywords:
Fixed point
Higher-order
Nonoscillatory solution

ABSTRACT

In this article, we study the existence of nonoscillatory solutions of higher order differential and delay differential equations with forcing term. Some new sufficient conditions are given. We use the Schauder's fixed point theorem to prove our results.

1. Introduction

The purpose of this article is to study higher-order differential and delay differential equations with forcing term of the form

$$\left[r(t)x^{(n)}(t)\right]^{(m)} + f(t, x(t)) = g(t), \tag{1}$$

$$\left[r(t)x^{(n)}(t)\right]^{(m)} + f(t, x(t-\tau)) = g(t) \tag{2}$$

and

$$\left[r(t)x^{(n)}(t)\right]^{(m)} + \int_a^b f(t, x(t-\xi))d\xi = g(t), \tag{3}$$

where $m \geqslant 1$ and $n \geqslant 2$ are integers, $f \in C([t_0, \infty) \times \mathbb{R}, \mathbb{R})$, $f(t, x)$ is non-decreasing or non-increasing function in x for each t, $xf(t, x) > 0$, $x \neq 0$, $\tau \geqslant 0$, $r \in C([t_0, \infty), (0, \infty))$, $g \in C([t_0, \infty), \mathbb{R})$ and $b > a \geqslant 0$. Here we give sufficient conditions for the existence of nonoscillatory solutions of (1)–(3).

The nonoscillatory behavior of solutions of differential and delay differential equations of the form (1)–(3) has been considered by different authors in the past. This work was motivated by the papers of Kusano and Naito [1,2] which are concerned with the existence of oscillatory and nonoscillatory solutions of fourth-order differential equation of the form

$$\left[r(t)y''(t)\right]'' + yF(y^2, t) = 0,$$

and Li and Fei [3] which is concerned with the existence of positive solutions of higher-order nonlinear delay differential equations of the form

$$\left(r(t)x^{(m-1)}(t)\right)' + f(t, x(t-\tau)) = 0.$$

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http://dx.doi.org/10.1016/j.aml.2014.08.010
0893-9659/

The existence of solutions of the differential equations and neutral differential equations also were investigated in [4] and [5,6], respectively. For related books, we refer the reader to [7–11].

By a solution of (1) we mean a function $x \in C([T, \infty), \mathbb{R})$, for some $T \geqslant t_0$, $x(t)$ is n times continuously differentiable and $r(t)x^{(n)}(t)$ is m times continuously differentiable on $[T, \infty)$ and such that (1) is satisfied for $t \geqslant T$. By a solution of (2) we mean a function $x \in C([t_1 - \tau, \infty), \mathbb{R})$, for some $t_1 \geqslant t_0$, $x(t)$ is n times continuously differentiable and $r(t)x^{(n)}(t)$ is m times continuously differentiable on $[t_1, \infty)$ and such that (2) is satisfied for $t \geqslant t_1$. By a solution of (3) we mean a function $x \in C([t_1 - b, \infty), \mathbb{R})$, for some $t_1 \geqslant t_0$, $x(t)$ is n times continuously differentiable and $r(t)x^{(n)}(t)$ is m times continuously differentiable on $[t_1, \infty)$ and such that (3) is satisfied for $t \geqslant t_1$.

2. Main results

Theorem 1. *Assume that*

$$\int_{t_0}^{\infty} \int_{s}^{\infty} \frac{s^{n-1}u^{m-1}}{r(s)} |f(u, c)| du ds < \infty \quad \text{for some } c \neq 0 \quad \text{and} \quad \int_{t_0}^{\infty} \int_{s}^{\infty} \frac{s^{n-1}u^{m-1}}{r(s)} |g(u)| du ds < \infty. \tag{4}$$

Then (1) has a bounded nonoscillatory solution.

Proof. Suppose (4) holds with $c > 0$, the case $c < 0$ can be treated similarly. Thus, one can choose a $t_1 \geqslant t_0$ sufficiently large such that

$$\frac{1}{(n-1)!(m-1)!} \int_{t}^{\infty} \int_{s}^{\infty} \frac{s^{n-1}u^{m-1}}{r(s)} [f(u, c) + |g(u)|]\, du ds \leqslant \alpha - M_1, \quad t \geqslant t_1, \tag{5}$$

where c is M_2 if $f(t, x)$ is non-decreasing in x for each t and c is M_1 if $f(t, x)$ is non-increasing in x for each t, M_1 and M_2 are defined in the following. Let Λ be the set of all continuous and bounded functions on $[t_1, \infty)$ with the sup norm. We define a closed, bounded and convex subset A of Λ as follows:

$$A = \{x \in \Lambda : M_1 \leqslant x(t) \leqslant M_2, \quad t \geqslant t_1\},$$

where M_1 and M_2 are positive constants such that

$$M_1 < \alpha < 2\alpha < M_1 + M_2.$$

Consider the operator $S : A \longrightarrow \Lambda$ defined by

$$(Sx)(t) = \alpha - \frac{(-1)^{m+n}}{(n-1)!(m-1)!} \int_{t}^{\infty} \int_{s}^{\infty} \frac{(s-t)^{n-1}(u-s)^{m-1}}{r(s)} [f(u, x(u)) - g(u)]\, du ds, \quad t \geqslant t_1.$$

Our goal is to show that S satisfies the assumptions of Schauder's fixed point theorem.

(i) S maps A into A. For $t \geqslant t_1$ and $x \in A$, using (5) we have

$$\begin{aligned}(Sx)(t) &\leqslant \alpha + \frac{1}{(n-1)!(m-1)!} \int_{t}^{\infty} \int_{s}^{\infty} \frac{(s-t)^{n-1}(u-s)^{m-1}}{r(s)} [f(u, c) + |g(u)|]\, du ds \\ &\leqslant \alpha + \frac{1}{(n-1)!(m-1)!} \int_{t}^{\infty} \int_{s}^{\infty} \frac{s^{n-1}u^{m-1}}{r(s)} [f(u, c) + |g(u)|]\, du ds \leqslant M_2\end{aligned}$$

and

$$\begin{aligned}(Sx)(t) &\geqslant \alpha - \frac{1}{(n-1)!(m-1)!} \int_{t}^{\infty} \int_{s}^{\infty} \frac{(s-t)^{n-1}(u-s)^{m-1}}{r(s)} [f(u, c) + |g(u)|]\, du ds \\ &\geqslant \alpha - \frac{1}{(n-1)!(m-1)!} \int_{t}^{\infty} \int_{s}^{\infty} \frac{s^{n-1}u^{m-1}}{r(s)} [f(u, c) + |g(u)|]\, du ds \geqslant M_1.\end{aligned}$$

Hence, S maps A into A.

(ii) S is continuous. Let $\{x_k\}$ be a convergent sequence of functions in A such that $x_k(t) \to x(t)$ as $k \to \infty$. Since A is closed, $x \in A$. For $t \geqslant t_1$,

$$\begin{aligned}|(Sx_k)(t) - (Sx)(t)| &\leqslant \frac{1}{(n-1)!(m-1)!} \int_{t}^{\infty} \int_{s}^{\infty} \frac{(s-t)^{n-1}(u-s)^{m-1}}{r(s)} |f(u, x_k(u)) - f(u, x(u))|\, du ds \\ &\leqslant \frac{1}{(n-1)!(m-1)!} \int_{t}^{\infty} \int_{s}^{\infty} \frac{s^{n-1}u^{m-1}}{r(s)} |f(u, x_k(u)) - f(u, x(u))|\, du ds.\end{aligned}$$

Since $|f(t, x_k(t)) - f(t, x(t))| \to 0$ as $k \to \infty$ by making use of the Lebesgue dominated convergence theorem, we see that

$$\lim_{k\to\infty} \|(Sx_k)(t) - (Sx)(t)\| = 0.$$

This means that S is continuous.