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On the concept and existence of solutions for fractional impulsive systems with Hadamard derivatives

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ABSTRACT

In this paper, a class of nonlinear fractional order differential impulsive systems with Hadamard derivative is discussed. First, a reasonable concept on the solutions of fractional impulsive Cauchy problems with Hadamard derivative and the corresponding fractional integral equations are established. Second, two fundamental existence results are presented by using standard fixed point methods. Finally, two examples are given to illustrate our theoretical results.

1. Introduction

Fractional calculus is a branch of mathematical analysis which discusses the differentiation and integration of arbitrary order and arises naturally in various areas of applied science and engineering. One can find more information on fractional calculus theory in Kilbas et al. [1] and application in controls in Baleanu et al. [2]. Fractional differential equations have attracted the attention of several researchers. In particular, the existence and uniqueness of the solution of Cauchy problems for fractional differential equations involving the Hadamard derivatives has been discussed in Kilbas et al. [1] in a non-sequential setting. Meanwhile, Klimek [3] investigated existence and uniqueness of the solution of sequential fractional differential equations with Hadamard derivative by using the contraction principle and a new, equivalent norm and metric. Very recently, Wang et al. [4] discussed the existence, blowing-up solutions and Ulam–Hyers stability of fractional differential equations with Hadamard derivative by using some classical methods. Further, Ahmad and Ntouyas [5] and Ma et al. [6] studied two dimensional fractional differential systems with Hadamard derivative.

In the present paper, we study fractional impulsive Cauchy problems of the form:

$$\begin{cases} {}_HD^{\alpha}_{1^+}u(t) = f(t, u(t)), \quad \alpha \in (0,1),\ t \in (1, e] \setminus \{t_1, t_2, \ldots, t_m\}, \\ \Delta u(t_i) = {}_HJ^{1-\alpha}_{1^+}u(t_i^+) - {}_HJ^{1-\alpha}_{1^+}u(t_i^-) = p_i, \quad p_i \in \mathbb{R},\ i = 1, 2, \ldots, m, \\ {}_HJ^{1-\alpha}_{1^+}u(1^+) = u_0, \quad u_0 \in \mathbb{R}, \end{cases} \tag{1}$$

where ${}_HD^{\alpha}_{1^+}$ denotes left-sided Hadamard fractional derivative of order α with the low limit 1 and ${}_HJ^{1-\alpha}_{1^+}$ denotes left-sided Hadamard fractional integral of order $1-\alpha$. The nonlinear term $f : J \times \mathbb{R} \to \mathbb{R}$ is a given function, and impulsive time sequences $\{t_i\}$ satisfy $1 = t_0 < t_1 < \cdots < t_m < t_{m+1} = e$. The symbol $\Delta u(t_i) = u(t_i^+) - u(t_i^-)$ where $u(t_i^+) = \lim_{\epsilon\to 0^+} u(t_i + \epsilon)$ and $u(t_i^-) = \lim_{\epsilon\to 0^-} u(t_i + \epsilon)$ represent the right and left limits of $u(t)$ at $t = t_i$, respectively.

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The rest of this paper is organized as follows. In Section 2, we give some notations for Hadamard fractional calculus, introduce a reasonable concept of a piecewise continuous solutions and establish an equivalent integral equation for the Hadamard fractional differential equations with impulse. In Section 3, we present two existence results by using the Banach contraction principle and Schauder's fixed point theorem. Finally, two examples are given to demonstrate the application of our main results.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts.

Definition 2.1 (*See [1, p. 110, (2.7.1)]*). The left-sided Hadamard fractional integral of order $\alpha \in \mathbb{R}^+$ of function $f(x)$ is defined by $({}_HJ^{\alpha}_{a+}f)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (\ln\frac{x}{t})^{\alpha-1}f(t)\frac{dt}{t}$, $(0 < a < x \le b)$, where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2 (*See [1, p. 111, (2.7.7)]*). The left-sided Hadamard fractional derivative of order $\alpha \in [n-1, n)$, $n \in \mathbb{Z}^+$ of function $f(x)$ is defined by $({}_HD^{\alpha}_{a+}f)(x) = \frac{1}{\Gamma(n-\alpha)}\left(x\frac{d}{dx}\right)^n \int_a^x \left(\ln\frac{x}{t}\right)^{n-\alpha+1} f(t)\frac{dt}{t}$, $(0 < a < x \le b)$, where $\Gamma(\cdot)$ is the Gamma function.

Let $0 < a < b < \infty$ and $C[a, b]$ be the Banach space of all continuous functions from $[a, b]$ into $\mathbb{R}$ with the norm $\|x\|_C = \max\{|x(t)| : t \in [a, b]\}$. For $0 \le \gamma < 1$, we denote the space $C_{\gamma,\ln}[a, b]$ by the weighted space of the continuous function f on the finite interval $[a, b]$, which is given by $C_{\gamma,\ln}[a, b] := \left\{f(x) : \left(\ln\frac{x}{a}\right)^{\gamma} f(x) \in C[a, b]\right\}$. Obviously, $C_{\gamma,\ln}[a, b]$ is the Banach space with the norm $\|f\|_{C_{\gamma,\ln}} = \|\left(\ln\frac{x}{a}\right)^{\gamma} f(x)\|_C$. Clearly, $C_{0,\ln}[a, b] := C[a, b]$. We denote the space $PC_{\gamma,\ln}[a, b]$ by the weighted space of the piecewise continuous function f on the finite interval $[a, b]$, which is given by $PC_{\gamma,\ln}[a, b] = \{f : \left(\ln\frac{t}{a}\right)^{\gamma} f$ is continuous at $t \in [a, b] \setminus \{t_1, t_2, \dots, t_m\}$, and $\left(\ln\frac{t}{a}\right)^{\gamma} f$ is continuous from left and has right hand limits at $t \in \{t_1, t_2, \dots, t_m\}\}$. Of course, $(PC_{\gamma,\ln}[a, b], \|\cdot\|_{PC_{\gamma,\ln}})$ is a Banach space endowed with the norm $\|f\|_{PC_{\gamma,\ln}} = \sup_{t\in[a,b]} \|\left(\ln\frac{t}{a}\right)^{\gamma} f(x)\|$. Moreover, $PC_{0,\ln}[a, b] := PC[a, b]$.

Let G be an open set in $\mathbb{R}$ and $f : [a, b] \times G \to \mathbb{R}$ be a function such that $f(x, y) \in C_{\gamma,\ln}[a, b]$ for any $y \in G$. Consider the following fractional Cauchy problem:

$$\begin{cases} {}_HD^{\alpha}_{a+}y(x) = f(x, y(x)), & (n-1 < \alpha \le n,\ x \in (a, b]), \\ {}_HD^{\alpha-k}_{a+}y(a^+) = b_k \in \mathbb{R}, & (k = 1, \dots, n;\ n = -[-\alpha]), \end{cases} \tag{2}$$

where ${}_HD^{\alpha-k}_{a+}y(a^+)$ means that the limit is taken at all points of the right-sided neighborhood $(a, a+\varepsilon)(\varepsilon > 0)$ of a.

A function $y \in C_{n-\alpha,\ln}[a, b]$ is said to be a solution of the fractional Cauchy problem (2) if y satisfies the equation ${}_HD^{\alpha}_{a+}y(x) = f(x, y(x))$ for each $x \in (a, b]$ and the conditions ${}_HD^{\alpha-k}_{a+}y(a^+) = b_k$, $k = 1, \dots, n$, $n = -[-\alpha]$.

Lemma 2.3 (*See [1, Theorem 3.28]*). *Let $\alpha > 0$, $n = -[-\alpha]$ and $0 \le \gamma < 1$. Let G be an open set in $\mathbb{R}$ and $f : (a, b] \times G \to \mathbb{R}$ be a function such that $f(x, y) \in C_{\gamma,\ln}[a, b]$ for any $y \in G$. A function $y \in C_{n-\alpha,\ln}[a, b]$ is a solution of the fractional integral equation*

$$y(x) = \sum_{j=1}^{n} \frac{b_j}{\Gamma(\alpha-j+1)}\left(\ln\frac{x}{a}\right)^{\alpha-j} + \frac{1}{\Gamma(\alpha)}\int_a^x \left(\ln\frac{x}{t}\right)^{\alpha-1} f(t, y(t))\frac{dt}{t}, \quad (0 < a < x),$$

if and only if y is a solution of the above fractional Cauchy problem (2).

In particular, a function $u \in C_{1-\alpha,\ln}[a, b]$ is a solution of the fractional Cauchy problem:

$$\begin{cases} ({}_HD^{\alpha}_{a+}u)(t) = f(t, u(t)), & 0 < \alpha < 1,\ t \in (a, b], \\ ({}_HJ^{1-\alpha}_{a+}u)(a^+) = u_0, & u_0 \in \mathbb{R}, \end{cases}$$

if and only if u is a solution of the following equation:

$$u(t) = \frac{u_0}{\Gamma(\alpha)}\left(\ln\frac{t}{a}\right)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_a^t \left(\ln\frac{t}{s}\right)^{\alpha-1} f(s, u(s))\frac{ds}{s}.$$

Next, we review some useful properties of the Hadamard type fractional integrals and derivatives which are used in the following sequels.

Lemma 2.4 (*See [1, Property 2.24]*). *If $\alpha > 0$, $\beta > 0$, and $0 < a < b < \infty$, then there exist*

$$\left({}_HJ^{\alpha}_{a+}\left(\ln\frac{t}{a}\right)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)\left(\ln\frac{x}{a}\right)^{\beta+\alpha-1}}{\Gamma(\beta+\alpha)}, \qquad \left({}_HD^{\alpha}_{a+}\left(\ln\frac{t}{a}\right)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)\left(\ln\frac{x}{a}\right)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}.$$

In particular, for $0 < \alpha < 1$, there exists $\left({}_HD^{\alpha}_{a+}\left(\ln\frac{t}{a}\right)^{\alpha-1}\right)(x) = 0$.