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Determination of the time-dependent perfusion coefficient in the bio-heat equation

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ABSTRACT

Identification of the time-dependent perfusion coefficient in the one-dimensional transient bio-heat conduction equation from Cauchy data is investigated. Necessary and sufficient conditions for the unique solvability of this inverse problem are provided.

1. Introduction

In this letter, we study the inverse coefficient identification problem which requires determining the time-dependent perfusion coefficient $q(t)$ and the temperature $T(x,t)$ satisfying the bio-heat equation, see Chan [1],

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} - q(t)T, \quad \text{in } (0,1) \times (0,t_f] := \Omega, \tag{1}$$

where $t_f > 0$ is an arbitrary time of interest, subject to the initial condition

$$T(x,0) = T_0(x), \quad x \in [0,1] \tag{2}$$

and the boundary conditions

$$T(0,t) = f(t), \quad t \in [0,t_f], \tag{3}$$

$$T(1,t) = g(t), \quad t \in [0,t_f], \tag{4}$$

$$-\frac{\partial T}{\partial x}(0,t) = h(t), \quad t \in [0,t_f]. \tag{5}$$

In the above mathematical formulation we assume that:

(A) $T_0 \in C^1[0,1], f, g, h \in C[0,t_f]$ are given functions satisfying the compatibility conditions

$$f(0) = T_0(0), \qquad g(0) = T_0(1), \qquad -h(0) = T_0'(0), \tag{6}$$

and we seek the solution $q \in C[0,t_f]$ and $T \in C^{2,1}(\overline{\Omega})$.

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We mention that related inverse problems in which the heat flux measurement (5) is replaced by an internal temperature or an internal mass measurement, as additional information, have been investigated elsewhere, see Cannon et al. [2] and Prilepko and Solov'ev [3]. On the other hand, the specification of the heat flux (5) formulates a more challenging inverse problem, see Pyatkov [4] and Lesnic [5]. In this paper, we report some further advances and methodology into the unique solvability of the inverse problem (1)–(5).

2. Analysis

The first step in the analysis employs the transformations

$$r(t) = \exp\left(\int_0^t q(\tau)d\tau\right), \qquad v(x,t) = r(t)T(x,t). \tag{7}$$

Note that $r \in C^1[0, t_f]$, $r > 0$, $r(0) = 1$ and

$$q(t) = \frac{r'(t)}{r(t)}, \qquad T(x,t) = \frac{v(x,t)}{r(t)}. \tag{8}$$

Under the transformations (7), the problem (1)–(5) linearizes as, see Lesnic [5],

$$\frac{\partial v}{\partial t}(x,t) = \frac{\partial^2 v}{\partial x^2}(x,t), \quad (x,t) \in \Omega, \tag{9}$$

$$v(x,0) = T_0(x), \quad x \in [0,1], \tag{10}$$

$$v(0,t) = r(t)f(t), \quad t \in [0,t_f], \tag{11}$$

$$v(1,t) = r(t)g(t), \quad t \in [0,t_f], \tag{12}$$

$$-\frac{\partial v}{\partial x}(0,t) = r(t)h(t), \quad t \in [0,t_f]. \tag{13}$$

First, remark that any of the conditions (11)–(13) applied at $t = 0$ and using (6), if non-zero, implies $r(0) = 1$. We can then write the solution of Eqs. (9), (10), (12) and (13) as, see Cannon [6],

$$v(x,t) = \int_0^1 G(x,t,\xi,0)T_0(\xi)d\xi + \int_0^t G(x,t,0,\tau)r(\tau)h(\tau)d\tau - \int_0^t G_\xi(x,t,1,\tau)r(\tau)g(\tau)d\tau, \quad (x,t) \in \overline{\Omega}, \tag{14}$$

where

$$G(x,t,\xi,\tau) = \frac{H(t-\tau)}{2\sqrt{\pi(t-\tau)}} \sum_{n=-\infty}^{\infty} (-1)^n \left[\exp\left(-\frac{(x-\xi+2n)^2}{4(t-\tau)}\right) + \exp\left(-\frac{(x+\xi+2n)^2}{4(t-\tau)}\right)\right], \tag{15}$$

is the Green function of the heat equation (9) with Neumann at $x = 0$ and Dirichlet at $x = 1$ homogeneous mixed boundary conditions. In (15), H denotes the Heaviside function.

Imposing (11) results in

$$r(t)f(t) = \int_0^1 G(0,t,\xi,0)T_0(\xi)d\xi + \int_0^t G(0,t,0,\tau)r(\tau)h(\tau)d\tau - \int_0^t G_\xi(0,t,1,\tau)r(\tau)g(\tau)d\tau, \quad t \in [0,t_f]. \tag{16}$$

Assuming that $f(t) > 0$ for $t \in [0, t_f]$, Eq. (16) can be rewritten as

$$r(t) = r_0(t) + \int_0^t K(t,\tau)r(\tau)d\tau, \quad t \in [0,t_f], \tag{17}$$

where

$$r_0(t) := \frac{1}{f(t)} \int_0^1 G(0,t,\xi,0)T_0(\xi)d\xi, \tag{18}$$

$$K(t,\tau) := \frac{1}{f(t)}[G(0,t,0,\tau)h(\tau) - G_\xi(0,t,1,\tau)g(\tau)]. \tag{19}$$

It is well-known, see Ivanchov ([7], p.12) that the Green function (15) satisfies

$$0 \le G(0,t,0,\tau) \le \frac{1}{\sqrt{\pi(t-\tau)}} + 1, \quad 0 \le \tau < t \le t_f, \tag{20}$$

$$0 \le -G_\xi(0,t,1,\tau) \le const., \quad 0 \le \tau < t \le t_f. \tag{21}$$