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# Note on improvements of order and efficiency in some iterative methods for solving nonlinear equations[☆]

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ABSTRACT

A generalization of two modified iterative schemes for solving nonlinear equations suggested by Traub and Ezquerro et al. is presented. This new technique allows to obtain local order of convergence $2\rho + 2$ from $\rho$ improving the above mentioned results where have been obtained $\rho+1$ and $2\rho+1$ from $\rho$ respectively. The efficiency is also improved. New numerical algorithms of third and fourth order are used to check the theoretical result given. They are illustrated with numerical examples.



## 1. Introduction

Many problems arising in diverse disciplines of engineering, sciences and nature can be described by a nonlinear equation of the following form [1]:

$$f(x) = 0, \tag{1}$$

where $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is a function sufficiently differentiable in a neighborhood $I$ of a simple root $\alpha$ of Eq. (1). If we are interested in approximating the root $\alpha$, we can do it by means of an iterative method like

$$x_{n+1} = \phi(x_n), \quad n \geq 0, \tag{2}$$

provided that the starting point $x_0$ is given. Two important features determine the choice of the previous iterative algorithm. One is the total number of iterations characterized by its local order of convergence, and the other is the computational cost that is measured by the necessary number of evaluations of the scalar function $f$ and its derivatives at each step. In the scalar case, these two features are linked by the efficiency index, $EI$, which is defined by $EI = \rho^{1/\omega}$, where $\rho$ is the local order of convergence and $\omega$ is the number of evaluations of $f$ and its derivatives that are needed per iteration to carry out (2) [2,3].

To improve the efficiency we can use some variants of the method as suggested in [4–9] and references therein. A well-known modification that improves the efficiency index of (2) is given by the following multipoint algorithm:

$$\begin{cases} x_0 \quad \text{given}, \\ z_n = \phi(x_n), \quad \text{(order of convergence } \rho) \\ x_{n+1} = z_n - f(z_n)/f'(x_n), \quad n \geq 0. \end{cases} \tag{3}$$

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It has the order of convergence $\rho + 1$ (see [3], Theorem 8-1, p. 166). Note that a particular case of (3) is the two step Newton's method [10,11]. Other situations can be found in [12].

Since (2) is a one-point iteration scheme with order of convergence $\rho$, then we need to evaluate the functions $f, f^{(1)}, f^{(2)}, \dots, f^{(\rho-1)}$ at each step as it is known. Its efficiency index is $EI = \rho^{1/\rho}$, whereas (3) has $EI = (\rho+1)^{1/(\rho+1)}$. So, if $\rho = 1, 2$ then the efficiency index of (3) is higher than that of (2).

A second modification, inspired in Chebyshev's method, that improves the efficiency index of (2) is given by the following multipoint algorithm:

$$\begin{cases} x_0 \quad \text{given}, \\ z_n = \phi(x_n), \quad \text{(order of convergence } \rho) \\ x_{n+1} = z_n - \big(1 + L_f(x_n, z_n)/2\big)\, u(z_n), \quad n \ge 0, \end{cases} \tag{4}$$

where $L_f(x_n, z_n) = f''(x_n)\, u(z_n)/f'(z_n)$ and $u(t) = f(t)/f'(t)$. Scheme (4) has order at least $2\rho+1$ [13]. As $\rho \ge 3$, $f(x_n)$, $f'(x_n)$ and $f''(x_n)$ have been already evaluated in the first step of the preceding algorithm. So, we have to evaluate two functions more, $f(z_n)$ and $f'(z_n)$, when (4) is applied. Hence, from the efficiency index $EI = \rho^{1/\rho}$ we obtain $EI = (2\rho+1)^{1/(\rho+2)}$.

## 2. Main result

Hereafter, we consider $\rho \ge 3$. Using the ideas presented in the previous section, we consider a modification of Newton's and Chebyshev's methods, that consists of using an appropriate operator in the second step of (3) and (4) respectively. We suggest the following variant:

$$\begin{cases} x_0 \quad \text{given}, \\ z_n = \phi(x_n), \quad \text{(order of convergence } \rho) \\ x_{n+1} = z_n - \big(1 + L_f(y_n, z_n)/2\big)\, u(z_n), \quad n \ge 0, \end{cases} \tag{5}$$

where $L_f(y_n, z_n) = f''(y_n)u(z_n)/f'(z_n)$. Note that in the second derivative of $f$ at $x_n$ in (4), we have substituted $x_n$ by Newton's point $y_n = x_n - u(x_n)$, obtaining $f''(y_n)$. Now, we can state and prove our main result.

**Theorem 2.1.** *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a function sufficiently differentiable in an open interval $I$. If $f$ has a simple root at $\alpha \in I$ and $x_0$ is sufficiently close to $\alpha$, then iterative method (5) has order of convergence at least $2\rho + 2$ ($\rho \ge 3$).*

**Proof.** From Taylor's formulae, we have

$$f(z_n) = f(\alpha + E_n) = f'(\alpha)\left(E_n + A_2E_n^2 + \mathcal{O}(E_n^3)\right),$$
$$f'(z_n) = f'(\alpha + E_n) = f'(\alpha)\left(1 + 2A_2E_n + \mathcal{O}(E_n^2)\right),$$

where $E_n = z_n - \alpha = C_0\, e_n^{\rho} + \mathcal{O}\left(e_n^{\rho+1}\right)$, $e_n = x_n - \alpha$ and $A_2 = f''(\alpha)/(2f'(\alpha))$. Developing $1/f'(z_n)$, $u(z_n)$ and $u(z_n)^2$ in powers of $E_n$, yields

$$\frac{1}{f'(z_n)} = \frac{1}{f'(\alpha)}\left(1 - 2A_2E_n + \mathcal{O}(E_n^2)\right),$$
$$u(z_n) = \left(E_n + A_2E_n^2 + \mathcal{O}(E_n^3)\right)\left(1 - 2A_2E_n + \mathcal{O}(E_n^2)\right) = E_n - A_2E_n^2 + \mathcal{O}(E_n^3). \tag{6}$$

From $f(y_n) = f'(\alpha)\left(\varepsilon_n + A_2\varepsilon_n^2 + A_3\varepsilon_n^3 + \mathcal{O}(\varepsilon_n^4)\right)$, where $\varepsilon_n = y_n - \alpha = A_2\, e_n^2 + \mathcal{O}(e_n^2)$, we have

$$f''(y_n) = f'(\alpha)\left(2\,A_2 + 6\,A_2\,A_3\,e_n^2 + \mathcal{O}(e_n^3)\right), \quad \text{with } A_3 = f'''(\alpha)/(6f'(\alpha)).$$

Furthermore,

$$L_f(y_n, z_n) = \frac{f''(y_n)\,u(z_n)}{f'(z_n)} = 2\left(A_2 + 3\,A_2\,A_3e_n^2\right)E_n - 6\left(A_2^2 + 3\,A_2^2\,A_3\,e_n^2\right)E_n^2 + \mathcal{O}(E_n^3). \tag{7}$$

Subtracting $\alpha$ from both sides of the second step of algorithm (5) and taking into account (6) and (7), the error is given by

$$|e_{n+1}| = |E_n - u(z_n) - L_f(y_n, z_n)u(z_n)/2| = 3|A_2\,A_3|e_n^2\,E_n^2 + \mathcal{O}(e_n^{2\,\rho+3}), \tag{8}$$

and the iterative method (5) has order of convergence at least $2\rho + 2$. ■

Since the iterative scheme (5) uses two function evaluations, $f(z_n)$ and $f'(z_n)$, more than the iterative method (2), the efficiency index of (5) is $(2\rho+2)^{1/(\rho+2)}$. So, we conclude that the efficiency index of (5) is better than the ones of (2)–(4).