Applied Mathematics Letters 33 (2014) 35–39

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

On the analytic integrability of the cored galactic Hamiltonian

Jaume Llibre [a,*], Claudia Valls [b]

[a] *Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain*

[b] *Departamento de Matemática, Instituto Superior Técnico, Universidade Técnica de Lisboa, Av. Rovisco Pais 1049–001, Lisboa, Portugal*

ARTICLE INFO

Article history:
Received 20 January 2014
Received in revised form 1 March 2014
Accepted 1 March 2014
Available online 12 March 2014

Keywords:
Analytic first integrals
Cored galactic Hamiltonian

ABSTRACT

We provide a complete characterization of the analytic first integrals of the cored galactic Hamiltonian.

1. Introduction and statement of the main results

We consider the planar cored galactic Hamiltonian

$$H = \frac{1}{2}(p_x^2 + p_y^2/q) + \sqrt{1 + x^2 + \frac{y^2}{q^2}}, \tag{1}$$

where $q > 0$. Its associated Hamiltonian system is

$$\begin{aligned} x' &= p_x, \\ y' &= \frac{p_y}{q}, \\ p_x' &= -\frac{x}{\sqrt{1 + x^2 + y^2/q^2}}, \\ p_y' &= -\frac{y}{q\sqrt{1 + x^2 + y^2/q^2}}, \end{aligned} \tag{2}$$

where the prime denotes derivative with respect to time t. Note that this Hamiltonian system has two degrees of freedom.

The potential

$$\sqrt{1 + x^2 + \frac{y^2}{q^2}}$$

has an absolute minimum and a reflection symmetry with respect to the two axes x and y. The motivation for the choice of these symmetries comes from the interest of this potential in galactic dynamics, see for instance [1–11]. The parameter q

* Corresponding author. Tel.: +34 935811303; fax: +34 935812790.
E-mail addresses: jllibre@mat.uab.cat (J. Llibre), cvalls@math.ist.utl.pt (C. Valls).

http://dx.doi.org/10.1016/j.aml.2014.03.001

gives the ellipticity of the potential, which ranges in the interval $0.6 \leq q \leq 1$. Lower values of q have no physical meaning and greater values of q are equivalent to reversing the role of the coordinate axes.

We note that the initial cored galactic Hamiltonian is

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \sqrt{1 + x^2 + \frac{y^2}{q^2}},$$

instead of the Hamiltonian (1) here studied. But in [1,8] and in other papers, they scale the variable p_y by the new variable $\sqrt{q}\, p_y$ obtaining the Hamiltonian (1) here considered. The reason for this scaling is that when Hamiltonian (1) is studied using normal forms, the zero order (unperturbed) Hamiltonian is

$$H_0 = \frac{1}{2}(p_x^2 + x^2) + \frac{1}{2q}(p_y^2 + y^2),$$

having frequencies 1 and $1/q$.

The main aim of this paper is to study the existence of a second analytic first integral F of the cored galactic Hamiltonian system (2) *independent* of H, i.e. the gradients of F and H are linearly independent at any point of the phase space except perhaps in a zero Lebesgue measure set. The existence of a such second independent first integral allows to describe completely the dynamics of a Hamiltonian system with two degrees of freedom, see for instance [12].

The Hamiltonian vector field X associated to system (2) is

$$X = p_x \frac{\partial}{\partial x} + \frac{p_y}{q}\frac{\partial}{\partial y} - \frac{x}{\sqrt{1 + x^2 + y^2/q^2}}\frac{\partial}{\partial p_x} - \frac{y}{q\sqrt{1 + x^2 + y^2/q^2}}\frac{\partial}{\partial p_y}.$$

Let U be an open and dense set in $\mathbb{R}^4$. We say that the non-constant function $F\colon U \to \mathbb{R}$ is a first integral of the vector field X on U, if $F(x(t), y(t), p_x(t), p_y(t)) =$ a constant for all values of t for which the solution $(x(t), y(t), p_x(t), p_y(t))$ of X is defined on U. Clearly F is a *first integral* of X on U if and only if $XF = 0$ on U. An *analytic* first integral is a first integral F being F an analytic function. Of course the Hamiltonian H is an analytic first integral of system (2). By definition a Hamiltonian system with 2 degrees of freedom having 2 independent first integrals is *completely integrable*, see for more details [12].

Proposition 1. *When $q = 1$ the cored galactic Hamiltonian system (2) is completely integrable with the first integrals H and $F = yp_x - xp_y$.*

Since $XF = 0$ when $q = 1$, and clearly H and F are independent the proposition follows. Hence, from now on we will restrict to the case $q \neq 1$.

Note that the cored galactic Hamiltonian system (2) is not a polynomial differential system, and consequently the Darboux theory of integrability (see for instance [13,14]), which is very useful for finding first integrals, cannot be applied to system (2).

Theorem 2. *The unique analytic first integrals of the cored galactic Hamiltonian system (2) with $q \neq 1$ are analytic functions in the variable H.*

The proof of Theorem 2 is given in Section 2. As a corollary we have the following result.

Corollary 3. *The cored galactic Hamiltonian system (2) with $q \neq 1$ is not completely integrable with analytic first integrals.*

2. Proof of Theorem 2

Doing the change of the independent variable from t to s given by

$$dt = \sqrt{1 + x^2 + y^2/q^2}\, ds,$$

the differential system (2) becomes

$$\begin{aligned}
\dot{x} &= p_x\sqrt{1 + x^2 + y^2/q^2},\\
\dot{y} &= \frac{p_y}{q}\sqrt{1 + x^2 + y^2/q^2},\\
\dot{p}_x &= -x,\\
\dot{p}_y &= -\frac{y}{q},
\end{aligned}$$

where the dot denotes the derivative with respect to the variable s. Taking the notation $Y = y/q$, $Q = 1/q^2 > 0$ we get

$$\begin{aligned}
\dot{x} &= p_x\sqrt{1 + x^2 + Y^2},\\
\dot{Y} &= Qp_y\sqrt{1 + x^2 + Y^2},\\
\dot{p}_x &= -x,\\
\dot{p}_y &= -Y.
\end{aligned}$$