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Concentration of homoclinic solutions for some fourth-order equations with sublinear indefinite nonlinearities[☆]

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ABSTRACT

In this paper, we mainly explore the phenomenon of concentration of homoclinic solutions for a class of nonperiodic fourth-order equations with sublinear indefinite nonlinearities. The proof is based on variational methods.

1. Introduction

In this paper, we consider a class of nonperiodic fourth-order equations with a parameter:

$$u^{(4)} + wu'' + \lambda a(x)u = f(x, u), \quad x \in \mathbb{R}, \tag{1.1}$$

where w is a constant, $\lambda > 0$ is a parameter, $f \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ and the function a satisfies the following conditions:

$(V1)$ $a \in C(\mathbb{R}, \mathbb{R})$ and $a \geq 0$ on $\mathbb{R}$; there exists $c > 0$ such that the set $\{a < c\} = \{x \in \mathbb{R} \mid a(x) < c\}$ is nonempty and $|\{a < c\}| < c_0 S_\infty^{-2}$, where $|\cdot|$ is the Lebesgue measure, S_∞ is the best Sobolev constant for the imbedding of $H^2(\mathbb{R})$ in $L^\infty(\mathbb{R})$ and c_0 is given in Lemma 2.1;

$(V2)$ $T = int\ a^{-1}(0)$ is nonempty and $\overline{T} = a^{-1}(0)$ such that T is a finite interval.

As usual, we say that a solution $u(x)$ of Eq. (1.1) is homoclinic (to 0) if $u(x) \to 0$ as $x \to \pm\infty$. In addition, if $u(x) \not\equiv 0$, then $u(x)$ is called a nontrivial homoclinic solution.

The above Eq. (1.1) has been put forward as the mathematical model for the study of pattern formation in physics and mechanics. For example, the well-known Extended Fisher–Kolmogorov (EFK) equation proposed by Coullet et al. in 1987 [1]

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in the study of phase transitions, as well as the Swift–Hohenberg (SH) equation which is a general model for the pattern-forming process derived in [2] to describe random thermal fluctuations in the Boussinesq equation and in the propagation of lasers in [3].

In recent years, the study of homoclinic and heteroclinic solutions for the fourth-order differential equations has begun to attract much attention; see [4–8]. These works mainly include the autonomous case and the nonautonomous case. More precisely, using the Mountain Pass Theorem, Tersian and Chaparova [7] obtained the existence of one nontrivial homoclinic solution for Eq. (1.1) with $\lambda = 1$ and $f(x, u) = c(x)u^2 + d(x)u^3$ when $a(x)$, $c(x)$ and $d(x)$ are continuous periodic functions and satisfy some suitable assumptions. Later, Li [4] considered the nonperiodic case of this equation and proved the existence of nontrivial homoclinic solution. Very recently, Sun and Wu [5] improved and extended the results in [4] and proved the existence of multiple homoclinic solutions for a class of nonperiodic fourth-order equations with a perturbation.

In order to obtain an important inequality and prove a compactness lemma in the above papers [4,5,7], the following assumption on the function a is necessary:

($\mathcal{A}$) there exist a positive constant a_1 such that $0 < a_1 \le a(x) \to +\infty$, for all $|x| \to +\infty$.

However, there are many functions a not satisfying the condition ($\mathcal{A}$), such as some nonnegative functions. Inspired by the above facts, the aim of this paper is to consider this case and study the existence of homoclinic solutions for Eq. (1.1), depending on a parameter λ. Moreover, it is more important that we will explore the phenomenon of concentration of homoclinic solutions as $\lambda \to \infty$.

Now we state our main results.

Theorem 1.1. *Assume that the conditions $(V1)$–$(V2)$ hold and $w < 2$. In addition, we assume that f satisfy the following condition:*

($D1$) *there exist a constant $\gamma \in (1, 2)$ and a positive function $b \in L^p(\mathbb{R})$ with $p \in (1, \frac{2}{2-\gamma}]$ such that $|f(x, u)| \le b(x)|u|^{\gamma-1}$ for all $(x, u) \in \mathbb{R} \times \mathbb{R}$;*

($D2$) *there exist two constants $\eta, \delta > 0$ such that $|F(x, u)| \ge \eta|u|^\gamma$ for all $x \in T$ and $u \in \mathbb{R}$ with $|u| \le \delta$, where $F(x, u) = \int_0^u f(x, s)ds$.*

Then there exists $\Lambda_0 > 0$ such that for every $\lambda > \Lambda_0$, Eq. (1.1) has at least one homoclinic solution u_λ.

Theorem 1.2. *Assume that $w \le 0$. Let u_λ be the homoclinic solution obtained in Theorem 1.1. Then $u_\lambda \to u_0$ in $H^2(\mathbb{R})$, as $\lambda \to \infty$, where $u_0 \in H^2(T) \cap H_0^1(T)$ is a nontrivial solutions of*

$$\begin{cases} u^{(4)} + wu'' = f(x, u) & \text{in } T, \\ u = 0, & \text{on } \partial T. \end{cases} \tag{E^∞}$$

The remainder of this paper is organized as follows. In Section 2, some preliminary results are presented. In Section 3, we give the proof of our main results.

2. Variational setting and preliminaries

In this section, we give the variational setting for Eq. (1.1) and some preliminaries.

Lemma 2.1 (*[7, Lemma 8]*). *Assume that $w < 2$. Then there exists a constant $c_0 > 0$ such that*

$$\int_{\mathbb{R}} \left[u''(x)^2 - wu'(x)^2 + u(x)^2\right] dx \ge c_0\|u\|_{H^2}^2, \quad \text{for all } u \in H^2(\mathbb{R}), \tag{2.1}$$

where $\|u\|_{H^2} = \left(\int_{\mathbb{R}} \left[u''(x)^2 + u'(x)^2 + u(x)^2\right] dx\right)^{1/2}$ is the norm of the Sobolev space $H^2(\mathbb{R})$.

Let

$$X = \left\{u \in H^2(\mathbb{R}) \mid \int_{\mathbb{R}} \left[u''(x)^2 - wu'(x)^2 + a(x)u(x)^2\right] dx < +\infty\right\}$$

be equipped with the inner product and norm

$$(u, v) = \int_{\mathbb{R}} \left[u''(x)v''(x) - wu'(x)v'(x) + a(x)u(x)v(x)\right] dx$$

and corresponding norm $\|u\|^2 = (u, u)$. For $\lambda > 0$, we also need the following inner product and norm

$$(u, v)_\lambda = \int_{\mathbb{R}} \left[u''(x)v''(x) - wu'(x)v'(x) + \lambda a(x)u(x)v(x)\right] dx,$$