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A novel approach to exponential stability of nonlinear non-autonomous difference equations with variable delays

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ABSTRACT

In this paper, by using a novel approach, we first prove a new generalization of discrete-type Halanay inequality. Based on our new generalized inequality, a novel criterion for the exponential stability of a certain class of nonlinear non-autonomous difference equations is proposed. Numerical examples are given to illustrate the effectiveness of the obtained results.

1. Introduction

Time-delay systems naturally arise in modeling a wide range of phenomena in the vivid world [1,2]. These practical systems are usually in the form of nonlinear and/or non-autonomous continuous-time systems with time-varying delays. Today, with the dramatically development of computer-based computational techniques, difference equations are found to be much appropriate mathematical representations for computer simulation, experiment and computation, which play an important role in realistic applications [3]. By a discretization from continuous-time systems, discrete-time systems described by difference equations inherit the similar dynamical behavior of the continuous ones. Therefore, the problem of stability analysis for difference equations with delays has received extensive attention from researchers (see, e.g., [4–10] and the references therein).

In most of the reported results concerned with the stability of difference equations with delays, a widely used approach is the Lyapunov–Krasovskii functional method. However, it relies heavily on how to choose an appropriate Lyapunov functional candidate and this usually leads to serious difficulties, especially in regard to nonlinear non-autonomous equations. An other effective approach is the use of discrete-type inequalities such as Gronwall inequalities or Halanay inequalities [3,5–7,9].

In [7], Udpin and Niamsup proved the following new discrete Halanay inequality which was shown more general than the one given in [5].

Theorem 1.1 ([7]). *Let $r \in \mathbb{Z}_+$, $p, q_r \in \mathbb{R}_+$, $q_i \in \mathbb{R}_0^+$, $h_i \in \mathbb{Z}_+$, $i = 1, \dots, r$, where $0 = h_0 < h_1 < \cdots < h_r$ and $\sum_{i=0}^{r} q_i < p \le 1$. Let $(x_n)_{n\in\mathbb{Z}^{-h_r}}$ be a sequence of real numbers satisfying*

$$\Delta x_n \le -px_n + \sum_{i=0}^{r} q_i x_{n-h_i}, \quad n \in \mathbb{Z}^0. \tag{1.1}$$

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Then there exists $\lambda_0 \in (0, 1)$ *such that*

$$x_n \le \max\{0, x_0, x_{-1}, \dots, x_{-h_r}\}\lambda_0^n, \quad n \in \mathbb{Z}^0. \tag{1.2}$$

Based on this new discrete Halanay inequality, new global exponential stability conditions for the following nonlinear difference equation

$$\Delta x_n = -px_n + f(n, x_n, x_{n-h_1}, \dots, x_{n-h_r}), \quad n \in \mathbb{Z}_+, \tag{1.3}$$

where $p > 0$, $r \in \mathbb{Z}_+$, $h_i \in \mathbb{Z}_+$, $i = 1, 2, \dots, r$, have been derived. However, it should be noted that Theorem 1.1 cannot be applied to non-autonomous equations. More precisely, even if $p = p(n)$, $q_i = q_i(n)$ satisfy $\sum_{i=0}^r q_i(n) < p(n) \le 1$ for all n, the conclusion of Theorem 1.1 is not correct (see the Appendix in this paper). Moreover, (1.1) can be written as follows:

$$x_{n+1} \le p_0 x_n + \sum_{i=0}^r q_i x_{n-h_i}, \quad n \in \mathbb{Z}^0, \tag{1.4}$$

where $p_0 = 1 - p$. Note that, in this case we have $\delta := p_0 + \sum_{i=0}^r q_i \in (0, 1)$.

Recently, in [8], a more general class of (1.4) was considered

$$x(n+1) \le h(n)\left[P_0 x(n) + P_1 x(n - \tau_1(n)) + \cdots + P_r x(n - \tau_r(n)) + I\right], \tag{1.5}$$

where function $h(.) : \mathbb{Z}^0 \to \mathbb{R}$ and matrices $P_i \in \mathbb{R}_+^{m\times m}$, $i = 0, 1, 2, \dots, r$, satisfy $0 < h(n) \le 1$ for all n and $\rho\left(\sum_{i=0}^r P_i\right) < 1$ ($\rho(A)$ denotes the spectral radius of matrix $A \in \mathbb{R}^{m\times m}$). An improved time-varying Halanay-type inequality has been proved and sufficient conditions on global exponential stability, global attracting set and ultimately bounded for (1.5) have been proposed. However, for more convenience in comparing stability conditions, let $m = 1, I = 0$ then (1.5) is reduced to the form of (1.4) with variable coefficients which satisfy $h(n)\sum_{i=0}^r P_i \le \sum_{i=0}^r P_i =: \delta < 1$ for all n. It can be seen in the aforementioned works, a strictly restriction that is required is that the sum of coefficients is uniformly less than one. This makes the obtained results conservative. So far, the problem of stability analysis for nonlinear non-autonomous difference equations without this restriction has not fully been investigated which motivates the present study.

In this paper, by using a novel approach, we first prove a new generalization of discrete-type Halanay inequality of the form

$$x_{n+1} \le p_0(n)x_n + \sum_{k=1}^r p_k(n) x_{n-\tau_k(n)}, \quad n \in \mathbb{Z}^0, \tag{1.6}$$

where $r \in \mathbb{Z}_+$ is given, $p_k(n) \ge 0$, $n \in \mathbb{Z}^0$, $k = 0, 1, \dots, r$, are variable coefficients, $0 \le \tau_k(n) \le \tau_k \in \mathbb{Z}_+$ are variable delays. We then derive a new criterion for the exponential stability of a more general class of (1.3). Furthermore, the restriction that the sum of coefficients is uniformly less than one will be removed. The rest of this paper is organized as follows. Section 2 presents a few preliminaries involved in this paper. Our main results are presented in Section 3. Section 4 provides some numerical examples to illustrate the effectiveness of the obtained results. The paper ends with a conclusion and cited references.

2. Preliminaries

Throughout this paper, we let $\mathbb{Z}$ and $\mathbb{Z}_+$ denote the set of integers and positive integers, respectively. For $r \in \mathbb{Z}$, we denote $\mathbb{Z}^r = \{m \in \mathbb{Z} : m \ge r\}$ and for $r_1, r_2 \in \mathbb{Z}$, $r_1 < r_2$, we denote $\mathbb{Z}[r_1, r_2] = \{r_1, r_1 + 1, \dots, r_2\}$.

Besides (1.6), we also consider the following general non-autonomous difference equations:

$$x_{n+1} = f_n(x_n, x_{n-\tau_1(n)}, \dots, x_{n-\tau_r(n)}), \quad n \in \mathbb{Z}^0, \tag{2.1}$$

where $f_n : \mathbb{R}^{r+1} \longrightarrow \mathbb{R}$ is continuous function for each integer $n \ge 0$. Assume that

$$\left|f_n(u_0, u_1, \dots, u_r)\right| \le \sum_{k=0}^r p_k(n)|u_k|, \quad (u_0, \dots, u_k) \in \mathbb{R}^{r+1}. \tag{2.2}$$

For each initial string $(x_n)_{n\in\mathbb{Z}[-\tau,0]}$, $\tau = \max_{k\in\mathbb{Z}[1,r]} \tau_k$, Eq. (2.1) has a unique solution $(x_n)_{n\in\mathbb{Z}^{-\tau}}$.

Definition 2.1 ([7,9]). Eq. (2.1) is said to be globally exponentially stable (GES) if there exist positive constants δ, λ such that every solution $(x_n)_{n\in\mathbb{Z}^{-\tau}}$ of (2.1) satisfies

$$|x_n| \le \delta \max_{j\in\mathbb{Z}[-\tau,0]} |x_j| e^{-\lambda n}, \quad \forall n \in \mathbb{Z}^0.$$

In Definition 2.1, if the decay function λn is replaced by a general function $\sigma(n)$, then we say (2.1) is globally generalized exponentially stable.