Applied Mathematics Letters 38 (2014) 20–26

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Nonclassical diffusion equations on $\mathbb{R}^N$ with singularly oscillating external forces

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ARTICLE INFO

Article history:
Received 23 April 2014
Received in revised form 12 June 2014
Accepted 12 June 2014
Available online 1 July 2014

Keywords:
Nonclassical diffusion equation
Uniform attractor
Singularly oscillating force
Unbounded domain
Uniform boundedness
Upper semicontinuity

ABSTRACT

We consider for $\rho \in [0, 1)$ and $\varepsilon > 0$, the nonclassical diffusion equation on $\mathbb{R}^N (N \geq 3)$ with a singularly oscillating external force,

$$u_t - \Delta u_t - \Delta u + f(x, u) + \lambda u = g_0(x, t) + \varepsilon^{-\rho} g_1(t/\varepsilon),$$

together with equation

$$u_t - \Delta u_t - \Delta u + \lambda u + f(x, u) = g_0(x, t)$$

formally corresponding to the limiting case $\varepsilon = 0$. Under suitable assumptions on the external force, the uniform (w.r.t. ε) boundedness of the related uniform attractors $\mathcal{A}^\varepsilon$ is established as well as the convergence of the attractor $\mathcal{A}^\varepsilon$ of the first equation to the attractor $\mathcal{A}^0$ of the second one as $\varepsilon \to 0^+$.

1. Introduction

For $\rho \in [0, 1)$ and $\varepsilon > 0$, we consider the following non-autonomous nonclassical diffusion equation with a singularly oscillating external force on $\mathbb{R}^N$,

$$u_t - \Delta u_t - \Delta u + f(x, u) + \lambda u = g_0(x, t) + \varepsilon^{-\rho} g_1(x, t/\varepsilon), \qquad u|_{t=\tau} = u_\tau. \tag{1.1}$$

Along with (1.1), we consider the equation

$$u_t - \Delta u_t - \Delta u + f(x, u) + \lambda u = g_0(x, t), \qquad u|_{t=\tau} = u_\tau, \tag{1.2}$$

without by rapid and singular oscillations, which formally corresponds to $\varepsilon = 0$.

The nonclassical diffusion equation arises as a model to describe physical phenomena, such as non-Newtonian flows, soil mechanic, and heat conduction (see e.g. [1]). The existence and long-time behavior of solutions to problem (1.1) has been studied extensively in recent years, for both autonomous case [2–6] and non-autonomous case [7,8,3,9]. In this paper we will study the boundedness and the upper semicontinuity of uniform attractors for a family of processes associated to problem (1.1) in the case of unbounded domains, the nonlinearity of Sobolev type, and the external force g is a singularly oscillating non-autonomous one.

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http://dx.doi.org/10.1016/j.aml.2014.06.008
0893-9659/

To study problems (1.1) and (1.2), we assume the following conditions:

(H1) The continuous nonlinearity $f(x,u)$ satisfies

$$f(x,u)\,u \geq -\mu u^2 - \phi_1(x), \tag{1.3}$$

$$f'_u(x,u) \geq -m, \tag{1.4}$$

$$|f(x,u)| \leq C\left(\phi_2(x) + |u|^\alpha\right), \tag{1.5}$$

$$\liminf_{|u|\to\infty} \frac{uf(x,u) - \kappa F(x,u)}{u^2} \geq 0, \quad \text{for some } \kappa > 0, \tag{1.6}$$

$$\liminf_{|u|\to\infty} \frac{F(x,u)}{u^2} \geq 0, \tag{1.7}$$

where $\phi_1 \in L^1(\mathbb{R}^N)$, $\phi_2 \in L^{\frac{2N}{N+2}}(\mathbb{R}^N)$ are two nonnegative functions, $1 < \alpha \leq \frac{N+2}{N-2}$, $0 < \mu, m < \lambda$, and $F(x,u) = \int_0^u f(x,s)\,ds$;

(H2) The functions $g_0, g_1 \in L^2_b(\mathbb{R}; L^2(\mathbb{R}^N))$, the space of translation bounded functions in $L^2_{loc}(\mathbb{R}; L^2(\mathbb{R}^N))$, that is,

$$\|g_i\|^2_{L^2_b} = \sup_{t\in\mathbb{R}} \int_t^{t+1} \|g_i(s)\|^2_{L^2(\mathbb{R}^N)} ds = M_i^2 < \infty \quad (i = 0, 1). \tag{1.8}$$

Moreover, we also assume that $\partial_t g_i \in L^2_b(\mathbb{R}; L^2(\mathbb{R}^N))$, and

$$\lim_{k\to+\infty} \sup_{t\in\mathbb{R}} \int_t^{t+1} \int_{|x|\geq k} |g_i(s,x)|^2 dxds = 0 \quad (i = 0, 1).$$

Putting

$$g^\varepsilon(x,t) := \begin{cases} g_0(x,t) + \varepsilon^{-\rho} g_1(x, t/\varepsilon), & \varepsilon > 0, \\ g_0(x,t), & \varepsilon = 0. \end{cases}$$

It is easy to check that $g^\varepsilon \in L^2_b(\mathbb{R}; L^2(\mathbb{R}^N))$, and

$$\|g^\varepsilon\|_{L^2_b} \leq Q_\varepsilon := \begin{cases} M_0 + \sqrt{2} M_1 \varepsilon^{-\rho}, & \varepsilon > 0, \\ M_0, & \varepsilon = 0. \end{cases} \tag{1.9}$$

For $g \in L^2_b(\mathbb{R}; L^2(\mathbb{R}^N))$, we denote by $\mathcal{H}_w(g)$ the closure of the set $\{g(\cdot + h) | h \in \mathbb{R}\}$ in $L^2_b(\mathbb{R}; L^2(\mathbb{R}^N))$ with the weak topology. Recall that the kernel $\mathcal{K}$ of a process $\{U(t,\tau)\}$ acting on X consists of all bounded complete trajectories:

$$\mathcal{K} = \{u(\cdot) \mid U(t,\tau)u(\tau) = u(t), \text{dist}\,(u(t), u(0)) \leq C_u, \ \forall t \geq \tau, \ \tau \in \mathbb{R}\}.$$

For each $s \in \mathbb{R}$, the set $\mathcal{K}(s) = \{u(s) \mid u(\cdot) \in \mathcal{K}\}$ is said to be the kernel section at time s.

Under the conditions (H1)–(H2) above, we proved in [8] the following result.

Theorem 1.1. *Assume that conditions* (H1) *and* (H2) *hold. Then for any fixed nonnegative number* ε*, the family of processes* $\{U^\varepsilon_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_w(g^\varepsilon)}$ *generated by* (1.1) *possesses a uniform attractor* $\mathcal{A}^\varepsilon$ *in the space* $H^1(\mathbb{R}^N)$*. Moreover,*

$$\mathcal{A}^\varepsilon = \bigcup_{\sigma\in\mathcal{H}_w(g^\varepsilon)} \mathcal{K}^\varepsilon_\sigma(s), \quad \forall s \in \mathbb{R}. \tag{1.10}$$

In this paper, following the general lines of the approach used in [10], we will prove the following facts concerning the family $\mathcal{A}^\varepsilon$ of uniform attractors of the processes generated by (1.1) and (1.2):

(i) The family $\mathcal{A}^\varepsilon$ is uniformly (w.r.t. ε) bounded in $H^1(\mathbb{R}^N)$:

$$\sup_{\varepsilon\in[0,1]} \|\mathcal{A}^\varepsilon\|_{H^1(\mathbb{R}^N)} < \infty.$$

(ii) The attractor $\mathcal{A}^\varepsilon$ converges to $\mathcal{A}^0$ as $\varepsilon \to 0^+$ in the standard Hausdorff semi-distance in $H^1(\mathbb{R}^N)$:

$$\lim_{\varepsilon\to 0^+} \{\text{dist}_{H^1(\mathbb{R}^N)}(\mathcal{A}^\varepsilon, \mathcal{A}^0)\} = 0.$$

Throughout this paper, for brevity, we denote by $\|\cdot\|$, $(\cdot,\cdot)$ the norm and scalar product in $L^2(\mathbb{R}^N)$, respectively. We also denote by C a generic constant, which is different from line to line or even in a same line.