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An existence result for a class of quasilinear singular competitive elliptic systems

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ABSTRACT

In this paper we establish existence and regularity of positive solutions for a singular quasilinear elliptic system with competitive structure. The approach is based on comparison properties, a priori estimates and the Schauder's fixed point theorem.

1. Introduction

In the present paper we focus on the system of quasilinear elliptic equations

$$\begin{cases} -\Delta_p u = f_1(u, v) & \text{in } \Omega \\ -\Delta_q v = f_2(u, v) & \text{in } \Omega \\ u, v > 0 \quad \text{in } \Omega \\ u, v = 0 \quad \text{on } \partial\Omega \end{cases} \tag{1.1}$$

on a bounded domain $\Omega \subset \mathbb{R}^N$ with a $C^{1,\alpha}$ boundary $\partial\Omega$, $\alpha \in (0, 1)$, which exhibits a singularity at zero. Here Δ_p (resp. Δ_q) stands for the p-Laplacian (resp. q-Laplacian) differential operator on $W_0^{1,p}(\Omega)$ (resp. $W_0^{1,q}(\Omega)$) with $1 < p, q \leq N$. Related to system (1.1), we assume that $f_i : (0, +\infty) \times (0, +\infty) \to (0, +\infty)$, $i = 1, 2$, are continuous functions satisfying the growth conditions:

$$\begin{cases} m_1 s_1^{\alpha_1} s_2^{\beta_1} \leq f_1(s_1, s_2) \leq M_1 s_1^{\alpha_1} s_2^{\beta_1} \text{ for all } s_1, s_2 > 0, \text{ with } M_1, m_1 > 0 \\ \text{and } \alpha_1 \in \mathbb{R}, \beta_1 < 0 \text{ such that } |\alpha_1| - \beta_1 < \min(1, p-1), \end{cases} \tag{1.2}$$

$$\begin{cases} m_2 s_1^{\alpha_2} s_2^{\beta_2} \leq f_2(s_1, s_2) \leq M_2 s_1^{\alpha_2} s_2^{\beta_2} \text{ for all } s_1, s_2 > 0, \text{ with } M_2, m_2 > 0 \\ \text{and } \beta_2 \in \mathbb{R}, \alpha_2 < 0 \text{ such that } |\beta_2| - \alpha_2 < \min(1, q-1). \end{cases} \tag{1.3}$$

A basic feature of our setting is that the singularity in problem (1.1) comes out through a competitive structure of the nonlinearities $f_1(u, v)$ and $f_2(u, v)$. It is caused by the fact that β_1 and α_2 are negative (see (1.2) and (1.3)), which prevents f_1 and f_2 to be increasing with respect to v and u, respectively. Due to this, the sub-supersolution method is not directly

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applicable to system (1.1) without additional assumptions. We refer to [1] for an approach within the method of sub-supersolutions. Another existence result obtained under different hypotheses and by means of adequate truncations can be found in [2]. We also mention that the semilinear case in (1.1) (i.e. $p = q = 2$) was treated in [3,4] by essentially using the linearity of the principal part. It is worth pointing out that the complementary situation for system (1.1) with respect to our setting is the so-called cooperative structure, that is assuming to have positive numbers β_1 and α_2 in (1.2) and (1.3). This case has attracted much interest (see [5,6,1,7]).

Our goal is to establish the existence and regularity of (positive) solutions for problem (1.1). To this end we develop some comparison arguments, which allow us to get an auxiliary result that provides a priori estimates. In turn, these estimates enable us to obtain our main result by applying the Schauder's fixed point theorem to a fixed point problem associated to system (1.1). The rest of the paper is organized as follows. Section 2 contains the needed comparison properties. Section 3 presents our existence and regularity result.

2. Auxiliary result

Given $1 < p < +\infty$, the spaces $L^p(\Omega)$ and $W_0^{1,p}(\Omega)$ are endowed with the usual norms $\|u\|_p = (\int_\Omega |u|^p\,dx)^{1/p}$ and $\|u\|_{1,p} = (\int_\Omega |\nabla u|^p\,dx)^{1/p}$, respectively. In the sequel, corresponding to $1 < p < +\infty$, we denote $p' = \frac{p-1}{p}$. We will also utilize the spaces $C(\overline{\Omega})$ and $C_0^{1,\beta}(\overline{\Omega}) = \{u \in C^{1,\beta}(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\}$ with $\beta \in (0,1)$. We recall that $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ for all $u \in W_0^{1,p}(\Omega)$. For later use, we denote by $\lambda_{1,p}$ and $\lambda_{1,q}$ the first eigenvalue of $-\Delta_p$ on $W_0^{1,p}(\Omega)$ and of $-\Delta_q$ on $W_0^{1,q}(\Omega)$, respectively. Let $\phi_{1,p}$ be the normalized positive eigenfunction of $-\Delta_p$ corresponding to $\lambda_{1,p}$, that is $-\Delta_p\phi_{1,p} = \lambda_{1,p}\phi_{1,p}^{p-1}$ in Ω, $\phi_{1,p} = 0$ on $\partial\Omega$, with $\|\phi_{1,p}\|_p = 1$. Similarly, let $\phi_{1,q}$ be the normalized positive eigenfunction of $-\Delta_q$ corresponding to $\lambda_{1,q}$, that is $-\Delta_q\phi_{1,q} = \lambda_{1,q}\phi_{1,q}^{q-1}$ in Ω, $\phi_{1,q} = 0$ on $\partial\Omega$, with $\|\phi_{1,q}\|_q = 1$. The strong maximum principle ensures the existence of positive constants l_1 and l_2 such that

$$l_1\phi_{1,p}(x) \le \phi_{1,q}(x) \le l_2\phi_{1,p}(x) \quad \text{for all } x \in \Omega. \tag{2.1}$$

In what follows, we introduce some functions which are useful to get comparison results and a priori estimates for solutions of problem (1.1). Let w_1 and w_2 be the unique weak solutions of the problems

$$\begin{cases} -\Delta_p w_1 = w_1^{\beta_1} & \text{in } \Omega \\ w_1 > 0 & \text{in } \Omega \\ w_1 = 0 & \text{on } \partial\Omega \end{cases} \quad \text{and} \quad \begin{cases} -\Delta_q w_2 = w_2^{\alpha_2} & \text{in } \Omega \\ w_2 > 0 & \text{in } \Omega \\ w_2 = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

respectively, which are known to satisfy

$$c_0\phi_{1,p}(x) \le w_1(x) \le c_1\phi_{1,p}(x) \quad \text{and} \quad c_0'\phi_{1,q}(x) \le w_2(x) \le c_1'\phi_{1,q}(x), \tag{2.3}$$

with positive constants c_0, c_1, c_0', c_1' (see [8]). Consider now the functions z_1 and z_2 defined by

$$-\Delta_p z_1 = \begin{cases} w_1^{\beta_1} & \text{in } \Omega \setminus \overline{\Omega}_\delta \\ -w_1^{\beta_1} & \text{in } \Omega_\delta, \end{cases} \qquad z_1 = 0 \text{ on } \partial\Omega \tag{2.4}$$

and

$$-\Delta_q z_2 = \begin{cases} w_2^{\alpha_2} & \text{in } \Omega \setminus \overline{\Omega}_\delta \\ -w_2^{\alpha_2} & \text{in } \Omega_\delta, \end{cases} \qquad z_2 = 0 \text{ on } \partial\Omega, \tag{2.5}$$

where

$$\Omega_\delta = \{x \in \Omega : d(x, \partial\Omega) < \delta\}$$

with a fixed $\delta > 0$ sufficiently small. On the basis of (2.3), the Hardy–Sobolev inequality (see, e.g., [9, Lemma 2.3]) guarantees that the right-hand side of (2.4) and (2.5) belongs to $W^{-1,p'}(\Omega)$ and $W^{-1,q'}(\Omega)$, respectively. Consequently, the Minty–Browder theorem (see [10, Theorem V.15]) implies the existence and uniqueness of z_1 and z_2 in (2.4) and (2.5). Moreover, (2.2), (2.3), (2.4), (2.5), the monotonicity of the operators $-\Delta_p$ and $-\Delta_q$, and [11, Corollary 3.1] yield

$$\frac{c_0}{2}\phi_{1,p}(x) \le z_1(x) \le c_1\phi_{1,p}(x) \quad \text{and} \quad \frac{c_0'}{2}\phi_{1,q}(x) \le z_2(x) \le c_1'\phi_{1,q}(x) \quad \text{in } \Omega, \tag{2.6}$$

for a possibly smaller $\delta > 0$. For later use we denote

$$R = \max\left\{\max_{\overline{\Omega}} \phi_{1,p}(x), \max_{\overline{\Omega}} \phi_{1,q}(x)\right\},$$

and fix a constant $\mu = \mu(\delta) > 0$ such that

$$\phi_{1,p}(x),\ \phi_{1,q}(x) \ge \mu \quad \text{in } \Omega \setminus \Omega_\delta.$$