Applied Mathematics Letters 38 (2014) 61–66

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Periodic solution of second order impulsive delay differential systems via variational method[☆]

Dezhu Chen, Binxiang Dai [*]

School of Mathematics and Statistics, Central South University, Changsha, Hunan, 410083, PR China

ARTICLE INFO

Article history:
Received 26 May 2014
Received in revised form 30 June 2014
Accepted 30 June 2014
Available online 9 July 2014

Keywords:
Impulse
Delay
Periodic solutions
Variational methods

ABSTRACT

In this paper we use variational methods to investigate the existence of periodic solutions for some second order delay differential systems with impulsive effects. Some new methods and results are obtained.

1. Introduction

In this paper, we study the following second-order delay differential systems with impulsive conditions:

$$\begin{cases} \ddot{u}(t) + \lambda u(t-\pi) = -f(t, u(t-\pi)), & \text{for } t \in (t_{k-1}, t_k) \\ \Delta \dot{u}(t_k) = g_k(u(t_k - \pi)), \\ u(0) = u(2\pi), \qquad \dot{u}(0) = \dot{u}(2\pi), \end{cases} \tag{1.1}$$

where $k \in \mathbb{Z}$, $u \in \mathbb{R}^N$, $\lambda \in \mathbb{R}$ is a parameter; $g_k(u) = grad_u G_k(u)$, $G_k \in C^1(\mathbb{R}^N, \mathbb{R})$ for each $k \in \mathbb{Z}$ and there exist an $m \in \mathbb{N}$ such that $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = \pi$, $t_{k+m+1} = t_k + \pi$ and $g_{k+m+1} = g_k$ for all $k \in \mathbb{Z}$; $f(t, u)$ is π-periodic in t and $f(t, u) = grad_u F(t, u)$ satisfies the following assumption:

(A) $F(t, x)$ is measurable in t for $x \in \mathbb{R}^N$ and continuously differentiable in x for a.e. $t \in [0, 2\pi]$, and there exists $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b \in L^1(0, 2\pi; \mathbb{R}^+)$ such that

$$|F(t, x)| + |f(t, x)| \le a(|x|)b(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, 2\pi]$.

Recently, a wide variety of techniques, especially the variational methods, have been developed to study boundary value problem of second order impulsive differential equations (see [1–5]). However, there are few results on the existence of solutions for delay differential equations obtained directly by the variational methods (see [6,7]). Moreover, there are no papers on the impulsive delay differential equations via variational methods.

In the present paper, the main purpose is to study the existence of periodic solutions for the system (1.1) via some recent critical point theorems.

☆ Supported by the National Natural Science Foundation of China (11271371) and the Postgraduate research and innovation project of Hunan Province (CX2011B078).

* Corresponding author.
E-mail addresses: bxdai@mail.csu.edu.cn, bxdai@csu.edu.cn (B. Dai).

http://dx.doi.org/10.1016/j.aml.2014.06.019
0893-9659/

2. Preliminaries

Let $H^1 = \{u : \mathbb{R} \to \mathbb{R}^N : u$ is absolutely continuous, $\dot{u}(t) \in L^2((0, 2\pi), \mathbb{R}^N)$ and $u(t) = u(t + 2\pi)\}$ with the inner product

$$\langle u, v\rangle = \int_0^{2\pi} u(t)v(t)dt + \int_0^{2\pi} \dot{u}(t)\dot{v}(t)dt, \quad \forall\, u, v \in H^1.$$

The corresponding norm is defined by

$$\|u\| = \left(\int_0^{2\pi} |u(t)|^2 dt + \int_0^{2\pi} |\dot{u}(t)|^2 dt\right)^{\frac{1}{2}}, \quad \forall\, u \in H^1.$$

It is well known that H^1 is compactly embedded in $C[0, 2\pi]$. There are constant c such that for all $u \in H^1$,

$$\|u\|_{L^p} \leq c\|u\|.$$

Let L_i $(i = 0, 1)$ be the operator from H^1 to H^1 defined by the following form

$$(L_0 u)(v) = \int_0^{2\pi} \dot{u}(t + \pi)\dot{v}(t)dt, \tag{2.1}$$

$$(L_1 u)(v) = \int_0^{2\pi} -\lambda u(t)v(t)dt. \tag{2.2}$$

It is easy to see that L_0 and L_1 are bounded linear operators on H^1. Moreover, $L := L_0 + L_1$ is self-adjoint on H^1. Similar to [7], H^1 has an orthogonal decomposition $H^1 = H^+ \oplus H^- \oplus H^0$, where $H^0 = KerL$ is finitely dimensional, and H^+, H^- are L-invariant subspaces such that for some $\sigma > 0$,

$$\langle Lu, u\rangle \geq \sigma\|u\|^2, \quad \forall u \in H^+,$$

and

$$\langle Lu, u\rangle \leq -\sigma\|u\|^2, \quad \forall u \in H^-.$$

Set $K = \{1, 2, \ldots, 2m + 1\}$. We construct the functionals ϕ and ψ on H^1 as follows,

$$\phi(u) = \frac{1}{2}\int_0^{2\pi} [\dot{u}(t + \pi)\dot{u}(t) - \lambda u^2(t)]dt - \int_0^{2\pi} F(t, u(t))dt + \sum_{k\in K} G_k(u(t_k)),$$

$$\psi(u) = -\int_0^{2\pi} F(t, u(t))dt + \sum_{k\in K} G_k(u(t_k)).$$

Clearly,

$$\phi(u) = \frac{1}{2}\langle Lu, u\rangle + \psi(u). \tag{2.3}$$

Since F satisfies the assumption (A) and g_k are continuous, a standard argument show that ϕ is continuously differentiable and weakly lower semi-continuous on H^1 and $\phi'(u)$ is defined by

$$\langle\phi'(u), v\rangle = \int_0^{2\pi} [\dot{u}(t + \pi)\dot{v}(t) - \lambda u(t)v(t) - f(t, u(t))v(t)]dt + \sum_{k\in K} g_k(u(t_k))v(t_k)$$

for all $u, v \in H^1$.

Moreover, there is a one-to-one correspondence between critical points of ϕ and the classical solutions of system (1.1).

Next, we give the main lemma used in this paper.

Lemma 2.1 ([8]). *Let E be a real Banach space with $E = V \oplus X$, where V is finite-dimensional. Suppose $\phi \in C^1(E, \mathbb{R})$ satisfies (PS) condition, and*

(ϕ_1) *there are constants $\rho, \tau > 0$ such that $\phi|_{\partial B_\rho \cap X} \geq \tau$, and*
(ϕ_2) *there is $e \in \partial B_1 \cap X$ and $R > \rho$ such that if*

$$Q \equiv (\bar{B}_R \cap V) \oplus \{re|0 < r < R\}, \quad \text{then } \phi|_{\partial Q} \leq 0.$$

Then ϕ possesses a critical value $c \geq \tau$ which can be characterized as

$$c = \inf_{h\in\Gamma} \max_{u\in Q} \phi(h(u)),$$

where

$$\Gamma = \{h \in C(\bar{Q}, E) : h = id \text{ on } \partial Q\}.$$