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Successively iterative method for fractional differential equations with integral boundary conditions

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ABSTRACT

In this paper, we study the boundary value problem of a fractional differential equation with nonlocal integral boundary conditions. Two successively iterative sequences are constructed, the conditions for the existence of the nontrivial sign-changing solutions to the differential equation are established.

1. Introduction

In this paper, we consider the following nonlinear fractional differential equations with integral boundary conditions

$$\begin{cases} D_{0+}^{\alpha}u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = u''(0) = 0, \\ u(1) = \displaystyle\int_0^{\eta} u(s)ds, \end{cases} \tag{1.1}$$

where $3 < \alpha \leq 4, 0 < \eta < 1, 0 < \frac{\eta^\alpha}{\alpha} < 1$, and D_{0+}^{α} is the standard Riemann–Liouville fractional derivative of order α.

Fractional differential equations have been proved to be new and valuable tools in the modeling of many phenomena in various fields of engineering, physics, and economics. And the theory of fractional differential equations has become an important aspect of differential equations (see [1–3]). In recent years, the existence of solutions for fractional-order differential equations with integral boundary value conditions have been extensively studied by many authors, see [4–10] and the references therein. The monotone iteration scheme is an interesting and effective technique for investigating the existence of solutions of nonlinear problem. The use of this method can be found in [11–14]. In this paper, we apply a monotone iterative technique to prove the existence of nontrivial solutions of the problem (1.1). The study is based on Lemma 1 proved in [15]. Our method is different from [4,5], and the nonlinear term $f(t, u)$ may change sign on some set. We also proved the existence of positive solutions, and only require the local continuity and local monotonicity of function $f(t, u)$. We note that the construction of the monotone iterative schemes does not require the existence of lower and upper solutions for the boundary value problems that we will study and it starts off with known two simple functions.

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2. Preliminaries and lemmas

In this section, we present several lemmas that are useful for the proof of our main results. Let X be the Banach space with norm $\|\cdot\|$. A nonempty closed set $K \subset X$ is called cone if K satisfies the following conditions: (i) if $x, y \in K$, then $x+y \in K$; (ii) if $x \in K$, then $\mu x \in K$ for any $\mu \geq 0$; (iii) if $0 \neq x \in K$, then $-x \notin K$. The cone K is called normal, if there exists $\rho > 0$ such that

$$\| x_1 + x_2 \| \geq \rho, \quad \forall x_1,\ x_2 \in K, \quad \| x_1 \| = \| x_2 \| = 1.$$

Let $x_1,\ x_2 \in X$. We write $x_1 \ll x_2$, if $x_2 - x_1 \in K$. We call the set $[x_1, x_2] = \{x \in X : x_1 \ll x \ll x_2\}$ order interval in X. The operator $T : [x_1, x_2] \to X$ is called increasing if $T\bar{x} \ll T\tilde{x}$ for any $\tilde{x}, \bar{x} \in [x_1, x_2]$ and $\bar{x} \ll \tilde{x}$.

Lemma 2.1 ([15]). *Let X be a Banach space ordered by a normal cone $K \subset X$. Assume that $T : [x_1, x_2] \to X$ is a completely continuous increasing operator such that $x_1 \ll Tx_1$, $Tx_2 \ll x_2$. Then T has a minimal fixed point x_* and a maximal fixed point x^* such that $x_1 \ll x_* \ll x^* \ll x_2$. Moreover, $x_* = \lim_{n\to\infty} T^n x_1$, $x^* = \lim_{n\to\infty} T^n x_2$, where $\{T^n x_1\}_{n=1}^{\infty}$ is an increasing sequence, $\{T^n x_2\}_{n=1}^{\infty}$ is a decreasing sequence.*

Lemma 2.2 ([6]). *Given $y(t) \in C(0, 1) \cap L^1(0, 1)$. The problem*

$$\begin{cases} D_{0+}^{\alpha} u(t) + y(t) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = u''(0) = 0, \\ u(1) = \displaystyle\int_0^{\eta} u(s)ds, \end{cases} \tag{2.1}$$

where $3 < \alpha \leq 4,\ 0 < \eta < 1$, is equivalent to

$$u(t) = \int_0^1 G(t, s)y(s)ds,$$

and

$$G(t,s) = \frac{1}{p(0)\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-1} - \dfrac{(\eta-s)^{\alpha}}{\alpha} t^{\alpha-1} - \left(1 - \dfrac{\eta^{\alpha}}{\alpha}\right)(t-s)^{\alpha-1}, & 0 \leq s \leq t \leq 1,\ s \leq \eta; \\ t^{\alpha-1}(1-s)^{\alpha-1} - \left(1 - \dfrac{\eta^{\alpha}}{\alpha}\right)(t-s)^{\alpha-1}, & 0 \leq \eta \leq s \leq t \leq 1; \\ t^{\alpha-1}(1-s)^{\alpha-1} - \dfrac{(\eta-s)^{\alpha}}{\alpha} t^{\alpha-1}, & 0 \leq t \leq s \leq \eta \leq 1; \\ t^{\alpha-1}(1-s)^{\alpha-1}, & 0 \leq t \leq s \leq 1,\ \eta \leq s, \end{cases} \tag{2.2}$$

here, $p(s) = 1 - \frac{\eta^{\alpha}}{\alpha}(1-s)$, $G(t, s)$ is called the Green function of problem (2.1). Obviously, $G(t, s)$ is continuous on $[0, 1] \times [0, 1]$.

Lemma 2.3 ([6]). *The function $G(t, s)$ defined by (2.2) has the following properties:*

(i) *$G(t,s) \geq \frac{1-p(0)}{\Gamma(\alpha)p(0)} t^{\alpha-1} s(1-s)^{\alpha-1},\ \forall t,\ s \in [0, 1]$;*
(ii) *$G(t,s) \leq [\frac{\alpha-1}{\Gamma(\alpha)} + \frac{4\eta^{\alpha-1}}{p(0)\Gamma(\alpha+1)}] t^{\alpha-1}(1-s)^{\alpha-1},\ \forall t,\ s \in [0, 1]$;*
(iii) *$G(t,s) \leq [\frac{\alpha-1}{\Gamma(\alpha)} + \frac{4\eta^{\alpha-1}}{p(0)\Gamma(\alpha+1)}] s(1-s)^{\alpha-1},\ \forall t,\ s \in [0, 1]$;*
(iv) *$p(s) > 0$, and $p(s)$ is not decreasing on $[0, 1]$;*
(v) *$G(t, s) > 0,\ \forall t,\ s \in (0, 1)$.*

From Lemma 2.3, we obtain the following Lemma

Lemma 2.4. *The function $G(t, s)$ defined by (2.2) satisfies the inequality*

$$\sigma_1(s) t^{\alpha-1} \leq G(t,s) \leq \sigma_2(s) t^{\alpha-1}, \quad \text{for } t,\ s \in [0, 1],$$

where $\sigma_1(s) = \frac{\eta^{\alpha}}{\Gamma(\alpha)(\alpha-\eta^{\alpha})} s(1-s)^{\alpha-1}$, $\sigma_2(s) = \frac{(\alpha-1)(\alpha-\eta^{\alpha})+4\eta^{\alpha-1}}{\Gamma(\alpha)(\alpha-\eta^{\alpha})}(1-s)^{\alpha-1}$.

Lemma 2.5. *The function $G(t, s)$ defined by (2.2) is continuous and satisfies*

$$|G(t_2,s) - G(t_1,s)| \leq \max_{0 \leq s \leq 1} |G(t_2,s) - G(t_1,s)| \leq \frac{\alpha-1}{\Gamma(\alpha)p(0)} \left[(1-\eta)^{\alpha-1} + 2\left(1 - \frac{\eta^{\alpha}}{\alpha}\right)\right](t_2 - t_1),$$

for $0 \leq t_1 \leq t_2 \leq 1$.