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Existence of nontrivial solution for a 4-sublinear Schrödinger–Poisson system

Shaowei Chen *, Dawei Zhang

School of Mathematical Sciences, Huaqiao University, Quanzhou 362021, China

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ABSTRACT

We consider a Schrödinger–Poisson system in $\mathbb{R}^3$ with potential indefinite in sign and a general 4-sublinear nonlinearity. Its variational functional does not satisfy the Palais–Smale condition in general. We use a finite-dimensional approximation method to obtain a sequence of approximate solutions. Then we prove that these approximate solutions converge weakly to a nontrivial solution of this system.

1. Introduction and statement of results

In this paper, we consider the Schrödinger–Poisson system

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(x,u), & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2, & \text{in } \mathbb{R}^3. \end{cases} \tag{1.1}$$

For V and f, we assume

(v). $V \in C(\mathbb{R}^3)$ is a bounded function in $\mathbb{R}^3$. Consider the following increasing sequence $\lambda_1 \leq \lambda_2 \leq \cdots$ of minimax values defined by

$$\lambda_n = \inf_{V\in\mathcal{V}_n} \sup_{u\in V,\ u\neq 0} \frac{\int_{\mathbb{R}^3} |\nabla u|^2 + Vu^2 dx}{\int_{\mathbb{R}^3} u^2 dx}$$

where $\mathcal{V}_k$ denotes the family of k-dimensional subspaces of $C_0^\infty(\mathbb{R}^3)$. Denote

$$\lambda_\infty = \lim_{n\to\infty} \lambda_n.$$

Then λ_∞ is the bottom of the essential spectrum of $-\Delta + V$ if it is finite and for every $n \in \mathbb{N}$ the inequality $\lambda_n < \lambda_\infty$ implies λ_n is an eigenvalue of $-\Delta + V$ of finite multiplicity (see [1,2]).

We assume there exists $k \geq 1$ such that

$$\lambda_k < 0 < \lambda_{k+1}. \tag{1.2}$$

($\mathbf{f_1}$). $f \in C(\mathbb{R}^3 \times \mathbb{R})$ and there exist $p \in (2,6)$ and $C > 0$ such that for all $(x,t) \in \mathbb{R}^3 \times \mathbb{R}$,

$$|f(x,t)| \leq C(1 + |t|^{p-1}).$$

* Corresponding author. Tel.: +86 15059510687.
E-mail address: swchen6@163.com (S. Chen).

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($\mathbf{f_2}$). $f(x,t) = o(t)$ as $t \to 0$ uniformly in $x \in \mathbb{R}^3$.
($\mathbf{f_3}$). $\limsup_{|t|\to+\infty} \frac{F(x,t)}{t^4} \le 0$ uniformly in $x \in \mathbb{R}^3$.
($\mathbf{f_4}$). there exists $0 < h < \lambda_\infty$ such that $4F(x,t) \le tf(x,t) + ht^2$ for all $(x,t) \in \mathbb{R}^3 \times \mathbb{R}$.

By elementary computations, one can see that the function

$$f(u) = F'(u) \text{ with } F(u) = \beta u \ln(1 + |u|^3), \quad u \in \mathbb{R}$$

fulfills the conditions ($\mathbf{f_1}$)–($\mathbf{f_4}$) if β is a sufficiently small positive number.

Our main result reads as follows:

Theorem 1.1. *Suppose that* ($\mathbf{v}$), *and* ($\mathbf{f_1}$) − ($\mathbf{f_4}$) *are satisfied, then the problem* (1.1) *has a nontrivial solution.*

Problem (1.1) arises in quantum mechanics and is related to the study of the nonlinear Schrödinger equation for a particle in an electromagnetic field or the Hartree–Fock equation. For a more detailed physical background of the Schrödinger–Poisson, readers can refer to [3,4] and the references therein.

This system has attracted considerable research attention in the recent decade. Many mathematical studies have been devoted to the case $\inf_{\mathbb{R}^3} V > 0$. There are many results on existence, nonexistence or multiplicity of solutions for (1.1) under this case. One can refer to [5,6,3,7–17,4,18,19], and [20–22].

In a very recent paper [23], Chen and Liu studied the problem (1.1) in the case that the potential V is indefinite in sign. They assume that V satisfies

($\mathbf{v}'$) $V \in C(\mathbb{R}^3)$ is bounded from below and, $\mu(V^{-1}(-\infty, M]) < \infty$ for every $M > 0$, where μ is the Lebesgue measure on $\mathbb{R}^3$. Moreover, the operator $-\Delta + V$ has negative eigenvalues,

and f satisfies ($\mathbf{f_1}$), ($\mathbf{f_2}$) and the following two assumptions

($\mathbf{f_3'}$) . $\lim_{|t|\to\infty} \frac{F(x,t)}{t^4} = +\infty$ uniformly in $x \in \mathbb{R}^3$.
($\mathbf{f_4'}$) . there exists $h > 0$ such that $4F(x,t) \le tf(x,t) + ht^2$ for all $(x,t) \in \mathbb{R}^3 \times \mathbb{R}$.

They proved that under these assumptions, the variational functional related to (1.1) (see (2.1) of Section 2) satisfies the Palais–Smale condition (see, for example, [24]). Then they used the classical local linking theorem to obtain a nontrivial solution of (1.1). However, under our assumptions on V, the variational functional related to (1.1) does not satisfy the Palais–Smale condition in general. Therefore, the classical critical point theory, such as the linking theorems, cannot be applied directly. To overcome this difficulty, we restrict the variational functional in a sequence of increasing finite dimensional spaces of $H^1(\mathbb{R}^3)$ and prove that in every such subspace, the restricted functional achieves its minimizer. Finally and interestingly, we prove that these minimizers can converge weakly to a nontrivial critical point of the variational functional and then we get a nontrivial solution of (1.1).

2. Proof of Theorems 1.1

Let $H^1(\mathbb{R}^3)$ be the standard Sobolev space with norm $\|u\| = (\int_{\mathbb{R}^3}(|\nabla u|^2 + u^2)dx)^{1/2}$. For $u \in H^1(\mathbb{R}^3)$, it is well known (see, for example, Theorem 2.2.1 of [25]) that the Poisson equation

$$-\Delta\phi = u^2$$

has a unique solution

$$\phi(x) = \phi_u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|}dy \quad \text{in } \mathcal{D}^{1,2}(\mathbb{R}^3).$$

Let

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)u^2\right)dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 dx - \int_{\mathbb{R}^3} F(x,u)dx, \quad u \in H^1(\mathbb{R}^3) \tag{2.1}$$

where

$$F(t) = \int_0^t f(x,\tau)d\tau.$$

Under the assumptions ($\mathbf{v}$) and ($\mathbf{f_1}$)–($\mathbf{f_4}$), Φ is a C^1 functional in $H^1(\mathbb{R}^3)$. The derivative of Φ is given by

$$\langle \Phi'(u), v\rangle = \int_{\mathbb{R}^3}(\nabla u\nabla v + V(x)uv)dx + \int_{\mathbb{R}^3}\phi_u uv dx - \int_{\mathbb{R}^3} f(x,u)dx, \quad \forall u, v \in H^1(\mathbb{R}^3).$$

It is easy to see that if u is a critical point of Φ, then (u, ϕ_u) is a solution of (1.1).

Let Y_1 be the space spanned by the eigenfunctions with corresponding eigenvalues less than λ_∞. Let $\{\psi_1, \ldots, \psi_n, \ldots\}$ be an orthogonal normal basis of Y_1 with ψ_i the eigenfunctions. Let Y_2 be the orthogonal complement space of Y_1 in $X = H^1(\mathbb{R}^3)$ and $\{e_1, \ldots, e_n, \ldots\}$ be an orthogonal normal basis of Y_2. For every $n \in \mathbb{N}$, let $X_n = \text{span}\{\psi_1, \ldots, \psi_n, e_1, \ldots, e_n\}$ and $\Phi_n = \Phi|_{X_n}$, the restriction of Φ in X_n.