Applied Mathematics Letters 38 (2014) 149–154

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Infinitely many periodic solutions of ordinary differential equations ☆

Yong He [a], Bo Xu [b,*]

[a] Department of Mathematics and Physics, Chongqing University of Science and Techonology, Chongqing 401331, People's Republic of China

[b] Department of Tourism and Mathematics, Sichuan Tourism University, Chengdu 610100, Sichuan, People's Republic of China

ARTICLE INFO

Article history:
Received 11 June 2014
Received in revised form 20 July 2014
Accepted 20 July 2014
Available online 30 July 2014

Keywords:
Periodic solution
Sublinear
Oscillating
Local minimum
Critical points

ABSTRACT

In this paper we assume the nonlinearity to be sublinear and get infinitely many periodic solutions by minimax methods.

1. Introduction and main result

Consider the following nonlinear problem:

$$\begin{cases} -\ddot{u}(t) = f(t, u(t)) \quad \text{a.e. } t \in [0, T] \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0 \end{cases} \tag{1}$$

where $T > 0$ and $f : [0, T] \times R \longrightarrow R$ is a continuous function.

Under various conditions, it has been proved that problem (1) has at least one solution by the variational methods (see [1–5]). Suppose that the nonlinearity $f(t, x)$ is bounded, that is, there exists $g \in L^1(0, T; R^+)$ such that

$$|f(t, x)| \le g(t)$$

for all $x \in R$ and $t \in [0, T]$. J. Mawhin and M. Willem [1] proved the existence of one solution for problem (1) under the condition that

$$\int_0^T F(t, x)dt \to -\infty$$

☆ Supported by education commission of Sichuan province, China (No. 0519802).

* Corresponding author. Tel.: +86 8352865917.
E-mail address: mmxubo@sina.com (B. Xu).

http://dx.doi.org/10.1016/j.aml.2014.07.018
0893-9659/

as $|x| \to \infty$, or that

$$\int_0^T F(t,x)dt \to +\infty$$

as $|x| \to \infty$, where $F(t,x) = \int_0^x f(t,s)ds$ is the potential.

Recently, Tang [3] generalizes this condition to the case that f is sublinear, that is, there exist constants $C_1, C_2 > 0$ and $\alpha \in [0, 1[$, such that

$$|f(t,x)| \le C_1|x|^\alpha + C_2 \tag{2}$$

for all $x \in R$ and $t \in [0, T]$. Under the condition that

$$|x|^{-2\alpha} \int_0^T F(t,x)dt \to -\infty$$

as $|x| \to \infty$, or that

$$|x|^{-2\alpha} \int_0^T F(t,x)dt \to +\infty$$

as $|x| \to \infty$, Tang proves that there exists at least one periodic solution. In this paper, we consider that the potential oscillates between these two conditions and show that there are infinitely many solutions. Our main result is the following theorem.

Theorem 1. *Suppose that $f(t,x)$ satisfies (2). Assume further that*

$$\limsup_{x\to\pm\infty} \int_0^T F(t,x)dt = +\infty \tag{3}$$

and that

$$\liminf_{x\to\pm\infty} |x|^{-2\alpha} \int_0^T F(t,x)dt = -\infty. \tag{4}$$

Then,

(I) *there exists a sequence of solutions (u_n), each of them is a minimax type critical point of the functional φ, and $\varphi(u_n) \to +\infty$, as $n \to \infty$;*

(II) *there exists another sequence of solutions (u_n^*), each of them is a local minimum point of the functional φ, and $\varphi(u_n^*) \to -\infty$, as $n \to \infty$.*

Corollary 1. *Suppose that $f(t,x)$ satisfies (2). Assume further that*

$$\limsup_{x\to\pm\infty} |x|^{-2\alpha} \int_0^T F(t,x)dt = +\infty \tag{5}$$

and that

$$\liminf_{x\to\pm\infty} |x|^{-2\alpha} \int_0^T F(t,x)dt = -\infty. \tag{6}$$

Then,

(I) *there exists a sequence of solutions (u_n), each of them is a minimax type critical point of the functional φ, and $\varphi(u_n) \to +\infty$, as $n \to \infty$;*

(II) *there exists another sequence of solutions (u_n^*), each of them is a local minimum point of the functional φ, and $\varphi(u_n^*) \to -\infty$, as $n \to \infty$.*

Remark 1. This paper is motivated by [6] where the authors consider two point boundary value problems and assume the nonlinearity to be bounded. Theorem 1 gives new results for problem (1). Specially, when $f(t,x) = g(x) + h(t)$, problem (1) becomes the Duffing equation. In this case Theorem 1 also gives new solvability conditions for the Duffing equation. On the other hand, there are functions $F(t,x)$ satisfying Theorem 1 and not satisfying assumptions of the previous results. For example, let $\alpha = \frac{1}{2}$ and

$$F(t,x) = (\sin x)|x|^{\frac{3}{2}} + h(t)x$$

where $h \in L^1(0, T; R)$.